

SYMMETRY-INVARIANT CONSERVATION LAWS OF PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. A simple characterization of the action of symmetries on conservation laws of partial differential equations is studied by using the general method of conservation law multipliers. The main results are applied to several examples of physical interest, including the generalized Korteweg-de Vries equation, a non-Newtonian generalization of Burger's equation, the b -family of peakon equations, and the Navier-Stokes equations for compressible, viscous fluids in two dimensions.

1. INTRODUCTION

Conservation laws have several important uses in the study of partial differential equations (PDEs), especially for determining conserved quantities and constants of motion, detecting integrability and linearizations, finding potentials and nonlocally-related systems, as well as checking the accuracy of numerical solution methods.

For any PDE system of normal type, regardless of whether a Lagrangian exists, the conservation laws admitted by the system can be found by a direct method [1, 2, 3] which is computationally similar to Lie's method for finding the symmetries [4, 5, 6] admitted by the system. The method for conservation laws is based on the standard result that there is a direct correspondence between conservation laws and multipliers whenever a PDE system can be expressed in a solved form in terms of a set of leading derivatives. This is the same property used by Lie's method for symmetries. In particular, all multipliers can be found by solving a linear system of determining equations which consists of the adjoint of the symmetry determining equations [4, 1, 2, 3] plus additional equations analogous to Helmholtz conditions [1, 2, 3]. For each solution of this determining system, a corresponding conservation law can be obtained by various direct integration methods [6, 7, 8]. Consequently, the problem of finding all conservation laws is reduced to a kind of adjoint of the problem of finding all symmetries.

In the case when a PDE system has a Lagrangian formulation, the symmetry determining equations constitute a self-adjoint linear system, and the multiplier determining system becomes the same as the determining equations for variational symmetries [6, 7]. Hence, the conservation laws admitted by a Lagrangian PDE system can be found without the need

to use the Lagrangian (and without the awkward need to consider “gauge terms” in the definition of variational symmetries).

For any given PDE system, the admitted symmetries have a natural action on the admitted conservation laws [4, 9, 10]. This action allows conservation laws to be divided into symmetry equivalence classes, which can be used to define a generating subset (or a basis) [11] for the set of all conservation laws of the PDE system. More recent work [12] has explored the properties of conservation laws that are invariant under the action of a given set of admitted symmetries, or more generally are homogeneous (mapped into themselves) under this symmetry action. Such conservation laws have at least three interesting applications [12]. Firstly, each symmetry-homogeneous conservation law represents a one-dimensional invariant subspace in the set of all non-trivial conservation laws. This is a useful feature when a generating subset (or basis) is being sought. Secondly, any symmetry-invariant conservation law will reduce to a first integral for the ODE obtained by symmetry reduction of the given PDE system when symmetry-invariant solutions of the system are sought. This provides a direct reduction of order of the ODE. Thirdly, the determining equations for multipliers can be augmented by a symmetry-homogeneity condition, which allows symmetry-homogeneous conservation laws to be obtained in a direct way by solving the augmented determining system.

In the present paper, we further develop this work and apply it to several nonlinear PDE systems arising in a variety of physical applications, including water waves, fluid flow, and gas dynamics.

In Sec. 2, the multiplier method for finding the conservation laws of a given PDE system is reviewed, along with the action of symmetries on conservation laws. In Sec. 3, the main properties of symmetry-invariant conservation laws and symmetry-homogeneous conservation laws are derived for normal PDE systems in any number of variables, generalizing the results introduced in Ref.[12] for single PDEs with two independent variables. Examples and applications of the multiplier method and of the general results on symmetry properties for conservation laws are presented in Sec. 4. Finally, some concluding remarks are made in Sec. 5.

2. PRELIMINARIES

Consider an N th-order system of PDEs

$$G^a(t, x, u, \partial u, \dots, \partial^N u) = 0, \quad a = 1, \dots, M \quad (2.1)$$

with m dependent variables u^α , $\alpha = 1, \dots, m$, and $n + 1$ independent variables t, x^i , $i = 1, \dots, n$, where $\partial^k u$ denotes all k th order partial derivatives of u with respect to t and x^i . The PDE system (2.1) is *normal* if each PDE is expressed in a solved form for some leading derivative of u such that all other terms in the system contain neither the leading derivative nor its differential consequences. As seen in the examples in Sec. 4, all typical PDE systems arising in physical applications are normal.

A *conservation law* [4, 6] of a given PDE system (2.1) is a total space-time divergence expression that vanishes on the solution space \mathcal{E} of the system,

$$(D_t T(t, x, u, \partial u, \dots, \partial^r u) + D_i X^i(t, x, u, \partial u, \dots, \partial^r u))|_{\mathcal{E}} = 0 \quad (2.2)$$

where the function T is a conserved density and the vector function X^i is a spatial flux. The pair

$$(T, X^i) = \Phi \quad (2.3)$$

is called a *conserved current*. Throughout,

$$D_t = \partial/\partial t + u_t^\alpha \partial/\partial u^\alpha + u_{tx}^\alpha \partial/\partial u_{x^\alpha}^\alpha + u_{tt}^\alpha \partial/\partial u_t^\alpha + \dots \quad (2.4)$$

denotes a total t -derivative, and

$$D_i = \partial/\partial x^i + u_{x^i}^\alpha \partial/\partial u^\alpha + u_{tx^i}^\alpha \partial/\partial u_t^\alpha + u_{x^i x^j}^\alpha \partial/\partial u_{x^j}^\alpha + \dots \quad (2.5)$$

denotes a total x^i -derivative.

Every conservation law (2.2) can be integrated over any given spatial domain $\Omega \subseteq \mathbb{R}^n$

$$\frac{d}{dt} \int_{\Omega} T d^n x = - \int_{\partial\Omega} X^i dA_i \quad (2.6)$$

showing that the rate of change of the quantity

$$\mathcal{C}[u] = \int_{\Omega} T d^n x \quad (2.7)$$

is balanced by the net flux through the domain boundary $\partial\Omega$. Here dA_i denotes the area element multiplied by the unit outward normal vector on $\partial\Omega$.

Two conservation laws are physically equivalent if they give the same conserved quantity (2.7) up to boundary terms. This occurs iff their conserved densities differ by a total spatial divergence $D_i \Theta^i$ on the solution space \mathcal{E} , and correspondingly, their fluxes differ by a total time derivative $-D_t \Theta^i$ modulo a divergence-free vector $D_j \Gamma^{ij}$, where $\Theta^i(t, x, u, \partial u, \dots, \partial^r u)$ is some vector function and $\Gamma^{ij}(t, x, u, \partial u, \dots, \partial^r u)$ is some antisymmetric tensor function. A conservation law is thereby called *trivial* if

$$T|_{\mathcal{E}} = D_i \Theta^i, \quad X^i|_{\mathcal{E}} = -D_t \Theta^i + D_j \Gamma^{ij} \quad (2.8)$$

so then any two equivalent conservation laws differ by a trivial conservation law. For a given PDE system (2.1), the set of all non-trivial conservation laws (up to equivalence) forms a vector space on which the symmetries of the system have a natural action.

An *infinitesimal symmetry* [4, 5, 6] of a given PDE system (2.1) is a generator

$$\mathbf{X} = \tau(t, x, u, \partial u, \dots, \partial^r u) \partial/\partial t + \xi^i(t, x, u, \partial u, \dots, \partial^r u) \partial/\partial x^i + \eta^\alpha(t, x, u, \partial u, \dots, \partial^r u) \partial/\partial u^\alpha \quad (2.9)$$

whose prolongation $\text{pr}\mathbf{X}$ leaves invariant the PDE system,

$$\text{pr}\mathbf{X}(G^a)|_{\mathcal{E}} = 0. \quad (2.10)$$

When acting on solutions $u^\alpha(t, x)$ of the PDE system, any infinitesimal symmetry (2.9) is equivalent to a generator with the *characteristic form*

$$\hat{\mathbf{X}} = P^\alpha \partial/\partial u^\alpha, \quad P^\alpha = \eta^\alpha - \tau u_t^\alpha - \xi^i u_{x^i}^\alpha \quad (2.11)$$

where the characteristic functions η^α, τ, ξ^i are determined by

$$0 = \text{pr}\hat{\mathbf{X}}(G^a)|_{\mathcal{E}} = (\delta_P G)^a|_{\mathcal{E}}. \quad (2.12)$$

Throughout,

$$\delta_g f = \frac{\partial f}{\partial u^\alpha} g^\alpha + \frac{\partial f}{\partial u_t^\alpha} D_t g^\alpha + \frac{\partial f}{\partial u_{x^i}^\alpha} D_i g^\alpha + \frac{\partial f}{\partial u_{tt}^\alpha} D_t^2 g^\alpha + \frac{\partial f}{\partial u_{tx^i}^\alpha} D_t D_i g^\alpha + \frac{\partial f}{\partial u_{x^i x^j}^\alpha} D_i D_j g^\alpha + \dots \quad (2.13)$$

denotes the Frechet derivative with respect to u^α , for any differential functions $f(t, x, u, \partial u, \partial^2 u, \dots)$ and $g^\alpha(t, x, u, \partial u, \partial^2 u, \dots)$.

An infinitesimal symmetry of a given PDE system (2.1) is *trivial* if its action on the solution space \mathcal{E} of the system is trivial, $\hat{\mathbf{X}}u^\alpha = 0$ for all solutions $u^\alpha(t, x)$. This occurs iff $P^\alpha|_{\mathcal{E}} = 0$. The corresponding generator (2.9) of a trivial symmetry is thus given by $\mathbf{X}|_{\mathcal{E}} = \tau\partial/\partial t + \xi^i\partial/\partial x^i + (\tau u_t^\alpha + \xi^i u_{x^i}^\alpha)\partial/\partial u^\alpha$, which has the prolongation $\text{pr}\mathbf{X}|_{\mathcal{E}} = \tau D_t + \xi^i D_i$. Conversely, any generator of this form on the solution space \mathcal{E} determines a trivial symmetry.

The *differential order of an infinitesimal symmetry* is defined to be the maximal differential order among its characteristic functions $P^\alpha|_{\mathcal{E}}$ evaluated on the solution space \mathcal{E} . Point symmetries are singled out by having the characteristic form

$$P^\alpha = \eta^\alpha(t, x, u) - \tau(t, x, u)u_t^\alpha - \xi^i(t, x, u)u_{x^i}^\alpha \quad (2.14)$$

which generates a transformation group on (t, x^i, u^α) . Symmetries having a more general first-order characteristic form

$$P^\alpha = \hat{\eta}^\alpha(t, x, u, \partial u) \quad (2.15)$$

do not generate a transformation group [4, 5] unless the number of dependent variables is $m = 1$, in which case the transformation group is a contact symmetry acting on $(t, x^i, u, u_t, u_{x^i})$.

The action of an infinitesimal symmetry (2.9) on the set of conserved currents (2.3) is given by [4, 9, 10, 12]

$$T_{\mathbf{X}} = \text{pr}\mathbf{X}(T) + T D_i \xi^i - X^i D_i \tau, \quad X_{\mathbf{X}}^i = \text{pr}\mathbf{X}(X^i) + X^i(D_t \tau + D_i X^i) - X^j D_j \xi^i - T D_t \xi^i. \quad (2.16)$$

When the symmetry is expressed in the characteristic form (2.11), its action has the simple form

$$T_{\hat{\mathbf{X}}} = \text{pr}\hat{\mathbf{X}}(T) = \delta_P T, \quad X_{\hat{\mathbf{X}}}^i = \text{pr}\hat{\mathbf{X}}(X^i) = (\delta_P X)^i. \quad (2.17)$$

The conserved currents $(T_{\mathbf{X}}, X_{\mathbf{X}}^i)$ and $(T_{\hat{\mathbf{X}}}, X_{\hat{\mathbf{X}}}^i)$ are equivalent,

$$(T_{\hat{\mathbf{X}}} - T_{\mathbf{X}})|_{\mathcal{E}} = D_i \Theta^i, \quad (X_{\hat{\mathbf{X}}}^i - X_{\mathbf{X}}^i)|_{\mathcal{E}} = -D_t \Theta^i + D_j \Gamma^{ij}, \quad (2.18)$$

with

$$\Theta^i = \tau X^i - T \xi^i, \quad \Gamma^{ij} = \xi^i X^j - \xi^j X^i \quad (2.19)$$

which follows from the relation $\text{pr}\mathbf{X} - \text{pr}\hat{\mathbf{X}} = \tau D_t + \xi^i D_i$.

An important question is when does the symmetry action on a given conserved current produce a trivial conserved current? A simple necessary and sufficient condition can be formulated by using a characteristic (canonical) form [4, 1, 2, 3, 6] for conservation laws, based on the following standard results [4, 6].

Lemma 1. *If a differential function $f(t, x, u, \partial u, \partial^2 u, \dots)$ vanishes on the solution space \mathcal{E} of a given PDE system (2.1), then $f = R_f(G)$ holds identically, where*

$$R_f = R_f^{(0)} + R_f^{(1)} + R_f^{(2)} + \dots \quad (2.20)$$

is a linear differential operator, depending on f , with coefficients that are non-singular on \mathcal{E} whenever the PDE system is normal.

Lemma 2. *A differential function $f(t, x, u, \partial u, \partial^2 u, \dots)$ is a total space-time divergence $f = D_t A + D_i B^i$ for some functions $A(t, x, u, \partial u, \partial^2 u, \dots)$ and $B^i(t, x, u, \partial u, \partial^2 u, \dots)$ iff $E_u^\alpha(f) = 0$ holds identically, where*

$$E_u^\alpha = \partial/\partial u^\alpha - D_t \partial/\partial u_t^\alpha - D_i \partial/\partial u_{x^i}^\alpha + D_t^2 \partial/\partial u_{tt}^\alpha + D_t D_i \partial/\partial u_{tx^i}^\alpha + D_i D_j \partial/\partial u_{x^i x^j}^\alpha + \dots \quad (2.21)$$

is the Euler-Lagrange operator.

From Lemma 1, a conservation law $(D_t T + D_i X^i)|_{\mathcal{E}} = 0$ for a normal PDE system $G^a = 0$ can be expressed as a divergence identity

$$D_t T + D_i X^i = R_{\Phi}(G) \quad (2.22)$$

holding off of the solution space \mathcal{E} of the PDE system, where $u^\alpha(t, x)$ is an arbitrary (sufficiently smooth) function. In this identity, integration by parts on the terms in $R_{\Phi}(G)$ yields

$$D_t \tilde{T} + D_i \tilde{X}^i = Q_a G^a \quad (2.23)$$

with

$$\begin{aligned} \tilde{T} &= T + R_{\Phi}^{(1)}{}_a G^a + R_{\Phi}^{(2)}{}_a D_t G^a - (D_t R_{\Phi}^{(2)}{}_a) G^a + R_{\Phi}^{(2)i}{}_a D_i G^a + \dots, \\ \tilde{X}^i &= X^i + R_{\Phi}^{(1)i}{}_a G^a + R_{\Phi}^{(2)ij}{}_a D_j G^a - (D_j R_{\Phi}^{(2)ij}{}_a) G^a - R_{\Phi}^{(2)i}{}_a D_t G^a + \dots, \end{aligned} \quad (2.24)$$

and

$$Q_a = R_{\Phi}^{(0)}{}_a - D_t R_{\Phi}^{(1)}{}_a - D_i R_{\Phi}^{(1)i}{}_a + D_t^2 R_{\Phi}^{(2)}{}_a + D_t D_i R_{\Phi}^{(2)i}{}_a + D_i D_j R_{\Phi}^{(2)ij}{}_a + \dots. \quad (2.25)$$

On the solution space \mathcal{E} of the PDE system $G^a = 0$, note that $\tilde{T}|_{\mathcal{E}} = T$ and $\tilde{X}^i|_{\mathcal{E}} = X^i$ reduce to the conserved density and the flux in the conservation law $(D_t T + D_i X^i)|_{\mathcal{E}} = 0$, and hence $(D_t \tilde{T} + D_i \tilde{X}^i)|_{\mathcal{E}} = 0$ is an equivalent conservation law. The identity (2.23) is called the *characteristic equation* for the conservation law, and the set of functions (2.25) is called the *multiplier*. In general a set of functions $Q^a(t, x, u, \partial u, \partial^2 u, \dots, \partial^r u)$ will be a multiplier iff its summed product with the PDEs $G^a = 0$ has the form of a total space-time divergence.

From the characteristic equation (2.23), it is straightforward to prove [14] the following basic result, which underlies the generality of the multiplier method.

Theorem 1. *For normal PDE systems (2.1) with no differential identities, there is a one-to-one correspondence between conservation laws (up to equivalence) and multipliers evaluated on the solution space of the system.*

A generalized version of this correspondence can be shown to hold for normal PDE systems that possess differential identities, such as Maxwell's electromagnetic field equations, magnetohydrodynamic systems, and Einstein's gravitational field equations. (See Ref.[14].)

For any normal PDE system, the *differential order of a conservation law* is defined to be the smallest differential order among all equivalent conserved currents. A conservation law is said to be of *low order* if each derivative variable $\partial^k u^\alpha$ that appears in its multiplier is related to some leading derivative of u^α by differentiations with respect to t, x^i . (Note that, therefore, the differential order r of the multiplier must be strictly less than the differential order N of the PDE system.) As seen in the examples in Sec. 4, physically important conservation laws, such as energy and momentum, are always of low order, whereas higher order conservation laws are typically connected with integrability.

All conservation law multipliers for any normal PDE system can be determined from Lemma 2 applied to the characteristic equation (2.23). This yields

$$0 = E_{u^\alpha}(Q_a G^a) = (\delta_Q^* G)_\alpha + (\delta_G^* Q)_\alpha \quad (2.26)$$

which is required to hold identically. Here a star denotes the adjoint of the Frechet derivative with respect to u^α :

$$\begin{aligned}
(\delta_g^* f)_\alpha &= \frac{\partial f}{\partial u^\alpha} g - D_t \left(\frac{\partial f}{\partial u_t^\alpha} g \right) - D_i \left(\frac{\partial f}{\partial u_{tx}^\alpha} g \right) + D_t^2 \left(\frac{\partial f}{\partial u_{tt}^\alpha} g \right) \\
&\quad + D_t D_i \left(\frac{\partial f}{\partial u_{tx}^\alpha} g \right) + D_i D_j \left(\frac{\partial f}{\partial u_{xixj}^\alpha} g \right) + \cdots \\
&= E_{u^\alpha}(f)g - E_{u^\alpha}^{(1,t)}(f)D_t g - E_{u^\alpha}^{(1,i)}(f)D_i g + E_{u^\alpha}^{(2,tt)}(f)D_t^2 g \\
&\quad + E_{u^\alpha}^{(2,ti)}(f)D_t D_i g + E_{u^\alpha}^{(2,ij)}(f)D_i D_j g + \cdots
\end{aligned} \tag{2.27}$$

for any differential functions $f(t, x, u, \partial u, \partial^2 u, \dots)$ and $g(t, x, u, \partial u, \partial^2 u, \dots)$, where $E_{u^\alpha}^{(1,t)}$, $E_{u^\alpha}^{(1,i)}$, $E_{u^\alpha}^{(2,tt)}$, $E_{u^\alpha}^{(2,ti)}$, $E_{u^\alpha}^{(2,ij)}$, etc. denote higher Euler operators [4]. The determining equation (2.26) can be converted into a linear system of equations for Q_a by the following steps.

On the solution space \mathcal{E} of a given PDE system (2.1), the Frechet derivative operators $\delta_G|_{\mathcal{E}}$ and $\delta_G^*|_{\mathcal{E}}$ vanish. Thus, the determining equation (2.26) implies

$$(\delta_Q^* G)_\alpha|_{\mathcal{E}} = 0. \tag{2.28}$$

From Lemma 1, it then follows that Q_a satisfies the identity

$$(\delta_Q^* G)_\alpha = R_Q(G)_\alpha \tag{2.29}$$

for some linear differential operator

$$(R_Q)_\alpha = R_Q^{(0)}{}_{a\alpha} + R_Q^{(1)}{}_{a\alpha} D_t + R_Q^{(1)i}{}_{a\alpha} D_i + R_Q^{(2)}{}_{a\alpha} D_t^2 + R_Q^{(2)i}{}_{a\alpha} D_t D_i + R_Q^{(2)ij}{}_{a\alpha} D_i D_j + \cdots \tag{2.30}$$

whose coefficients are non-singular on \mathcal{E} as the PDE system $G^a = 0$ is assumed to be normal. Substitution of this identity (2.29) into the determining equation (2.26) yields

$$0 = R_Q(G)_\alpha + (\delta_G^* Q)_\alpha \tag{2.31}$$

where $u^\alpha(t, x)$ is an arbitrary (sufficiently smooth) function. This equation (2.31) can be split with respect to each leading derivative of u^α in the PDE system and each differential consequence of these derivatives. The splitting yields a linear system of equations

$$R_Q^{(0)}{}_{a\alpha} = E_{u^\alpha}(Q_a) \tag{2.32a}$$

$$R_Q^{(1)}{}_{a\alpha} = -E_{u^\alpha}^{(1,t)}(Q_a) \tag{2.32b}$$

$$R_Q^{(1)i}{}_{a\alpha} = -E_{u^\alpha}^{(1,i)}(Q_a) \tag{2.32c}$$

$$R_Q^{(2)}{}_{a\alpha} = E_{u^\alpha}^{(2,tt)}(Q_a) \tag{2.32d}$$

$$R_Q^{(2)i}{}_{a\alpha} = E_{u^\alpha}^{(2,ti)}(Q_a) \tag{2.32e}$$

$$R_Q^{(2)ij}{}_{a\alpha} = E_{u^\alpha}^{(2,ij)}(Q_a) \tag{2.32f}$$

$$\vdots \tag{2.32g}$$

Hence, we obtain the following determining system for conservation law multipliers.

Proposition 1. *The determining equation (2.26) for conservation law multipliers of a normal PDE system (2.1) is equivalent to the linear system of equations (2.28) and (2.32).*

The first equation (2.28) in this linear system is the adjoint of the symmetry determining equation (2.12), and its solutions $Q_a(t, x, u, \partial u, \partial^2 u, \dots)$ are called adjoint-symmetries [1, 2, 3]. The remaining equations (2.32) comprise Helmholtz conditions [4] which are necessary and sufficient for an adjoint-symmetry $Q_a(t, x, u, \partial u, \partial^2 u, \dots)$ to have the form (2.25) where $\Phi = (T, X^i)$ is a conserved current.

These equations can be solved computationally by the same standard procedure [4, 5, 6] used to solve the determining equation for symmetries. Moreover, the multiplier determining system is, in general, more overdetermined than is the symmetry determining equation, and hence the computation of multipliers is easier than the computation of symmetries.

This formulation of a determining system for conservation law multipliers has a simple adjoint relationship to Noether's theorem, as outlined in Ref.[12]. First, recall [4, 5, 6] that the condition for a PDE system (2.1) to be given by Euler-Lagrange equations

$$G^a = E_u \alpha(L) = 0 \quad (2.33)$$

for some Lagrangian $L(t, x, u, \partial u, \partial^2 u, \dots)$ is that

$$(\delta_f G)^a = (\delta_f^* G)_\alpha \quad (2.34)$$

holds for arbitrary differential functions $f^\alpha(t, x, u, \partial u, \partial^2 u, \dots)$. In particular, it is necessary that in the PDE system the number M of equations is the same as the number m of dependent variables, whereby the indices a and α can be identified.

The following result is now straightforward to establish (see Refs.[1, 2, 3, 12]).

Theorem 2. *For any normal PDE system (2.1), conservation law multipliers are adjoint-symmetries (2.28) that satisfy Helmholtz conditions (2.32). In the case when the PDE system is an Euler-Lagrange system (2.33), adjoint-symmetries are the same as symmetries, and the Helmholtz conditions are equivalent to symmetry invariance of the Lagrangian modulo a total space-time divergence, so multipliers for a Lagrangian PDE system consequently are the same as variational symmetries.*

In applications of Theorem 2, the use of a Lagrangian to obtain the conserved current from a variational symmetry is replaced by either a homotopy integral formula [4, 2, 3] or direct integration of the characteristic equation [13, 6], both of which are applicable for any normal PDE system. If a given PDE system possesses a scaling symmetry then any conserved current having non-zero scaling weight can be obtained from an algebraic formula [15] in terms of a multiplier (see also Refs.[16, 17]).

3. MAIN RESULTS

A simple expression for the action of an infinitesimal symmetry on a conservation law multiplier will now be presented. The following Frechet derivative identity [4, 18, 1, 2, 3, 6] is needed.

Lemma 3. *The Frechet derivative (2.13) and its adjoint (2.27) satisfy the identity*

$$h\delta_g f - g\delta_h^* f = D_t \Psi_f^t(g, h) + D_i \Psi_f^i(g, h) \quad (3.1)$$

with

$$\begin{aligned}
\Psi_f^t(g, h) = & g^\alpha \left(h \frac{\partial f}{\partial u_t^\alpha} - D_t \left(h \frac{\partial f}{\partial u_{tt}^\alpha} \right) - D_i \left(h \frac{\partial f}{\partial u_{tx^i}^\alpha} \right) + \dots \right) \\
& + D_t g^\alpha \left(h \frac{\partial f}{\partial u_{tt}^\alpha} - D_t \left(h \frac{\partial f}{\partial u_{ttt}^\alpha} \right) - D_i \left(h \frac{\partial f}{\partial u_{ttx^i}^\alpha} \right) + \dots \right) \\
& + D_t^2 g^\alpha \left(h \frac{\partial f}{\partial u_{ttt}^\alpha} - D_t \left(h \frac{\partial f}{\partial u_{tttt}^\alpha} \right) - D_i \left(h \frac{\partial f}{\partial u_{tttx^i}^\alpha} \right) + \dots \right) + \dots
\end{aligned} \tag{3.2}$$

modulo a trivial term $D_i \Theta^i$, and

$$\begin{aligned}
\Psi_f^i(g, h) = & g^\alpha \left(h \frac{\partial f}{\partial u_{x^i}^\alpha} - D_j \left(h \frac{\partial f}{\partial u_{x^i x^j}^\alpha} \right) + D_j D_k \left(h \frac{\partial f}{\partial u_{x^i x^j x^k}^\alpha} \right) + \dots \right) \\
& + D_t g^\alpha \left(h \frac{\partial f}{\partial u_{tx^i}^\alpha} - D_j \left(h \frac{\partial f}{\partial u_{tx^i x^j}^\alpha} \right) + D_j D_k \left(h \frac{\partial f}{\partial u_{tx^i x^j x^k}^\alpha} \right) + \dots \right) \\
& + D_j g^\alpha \left(h \frac{\partial f}{\partial u_{x^i x^j}^\alpha} - D_k \left(h \frac{\partial f}{\partial u_{x^i x^j x^k}^\alpha} \right) + D_k D_l \left(h \frac{\partial f}{\partial u_{x^i x^j x^k x^l}^\alpha} \right) + \dots \right) + \dots
\end{aligned} \tag{3.3}$$

modulo a trivial term $-D_t \Theta^i + D_j \Gamma^{ij}$, where $f(t, x, u, \partial u, \partial^2 u, \dots)$, $g^\alpha(t, x, u, \partial u, \partial^2 u, \dots)$ and $h(t, x, u, \partial u, \partial^2 u, \dots)$ are arbitrary differential functions.

As first shown in Ref.[18, 1], the identity (3.1) yields a conserved current when $f = G^a$, $g = P^\alpha$, $h = Q_a$, where P^α is the characteristic of an infinitesimal symmetry (2.11) and Q_a is an adjoint-symmetry, which satisfy $(\delta_P G)^a|_{\mathcal{E}} = 0$ and $(\delta_Q^* G)_\alpha|_{\mathcal{E}} = 0$. Note, by Lemma 1, it follows that

$$(\delta_P G)^a = R_P(G)^a, \quad (\delta_Q^* G)_\alpha = R_Q(G)_\alpha \tag{3.4}$$

holds identically, where R_Q is the linear differential operator (2.30), and R_P is a similar linear differential operator

$$(R_P)^a = R_P^{(0)a} + R_P^{(1)a} D_t + R_P^{(1)ai} D_i + R_P^{(2)a} D_t^2 + R_P^{(2)ai} D_t D_i + R_P^{(2)aij} D_i D_j + \dots \tag{3.5}$$

In both operators R_P and R_Q , the coefficients are non-singular on \mathcal{E} as the PDE system $G^a = 0$ is assumed to be normal.

We can now state the first main result for PDE systems with any number of variables, which extends the results derived in recent work [12] on single PDEs with two independent variables.

Theorem 3. *For a given normal PDE system (2.1), let $\Phi = (T, X^i)$ be a conserved current and Q_a be its multiplier, and let $\hat{\mathbf{X}} = P^\alpha \partial / \partial u^\alpha$ be an infinitesimal symmetry. Then $\Psi_G(P, Q) = (\Psi_G^t(P, Q), \Psi_G^i(P, Q))$ defines a conserved current which is equivalent to the conserved current obtained by the symmetry action $\Phi_{\hat{\mathbf{X}}} = (T_{\hat{\mathbf{X}}}, X_{\hat{\mathbf{X}}}^i) = \text{pr} \hat{\mathbf{X}}(\Phi)$ on Φ . In explicit*

form,

$$\begin{aligned}
T_{\hat{\mathbf{X}}} &= \delta_P T = \Psi_G^t(P, Q) \\
&= P^\alpha \left(Q_a \frac{\partial G^a}{\partial u_t^\alpha} - D_t \left(Q_a \frac{\partial G^a}{\partial u_{tt}^\alpha} \right) - D_i \left(Q_a \frac{\partial G^a}{\partial u_{tx^i}^\alpha} \right) + \dots \right) \\
&\quad + D_t P^\alpha \left(Q_a \frac{\partial G^a}{\partial u_{tt}^\alpha} - D_t \left(Q_a \frac{\partial G^a}{\partial u_{ttt}^\alpha} \right) - D_i \left(Q_a \frac{\partial G^a}{\partial u_{ttxi}^\alpha} \right) + \dots \right) \\
&\quad + D_t^2 P^\alpha \left(Q_a \frac{\partial G^a}{\partial u_{ttt}^\alpha} - D_t \left(Q_a \frac{\partial G^a}{\partial u_{tttt}^\alpha} \right) - D_i \left(Q_a \frac{\partial G^a}{\partial u_{tttxi}^\alpha} \right) + \dots \right) + \dots
\end{aligned} \tag{3.6}$$

modulo trivial terms $D_i \Theta^i$, and

$$\begin{aligned}
X_{\hat{\mathbf{X}}}^i &= (\delta_P X)^i = \Psi_G^i(P, Q) \\
&= P^\alpha \left(Q_a \frac{\partial G^a}{\partial u_{x^i}^\alpha} - D_j \left(Q_a \frac{\partial G^a}{\partial u_{x^i x^j}^\alpha} \right) + D_j D_k \left(Q_a \frac{\partial G^a}{\partial u_{x^i x^j x^k}^\alpha} \right) + \dots \right) \\
&\quad + D_t P^\alpha \left(Q_a \frac{\partial G^a}{\partial u_{tx^i}^\alpha} - D_j \left(Q_a \frac{\partial G^a}{\partial u_{tx^i x^j}^\alpha} \right) + D_j D_k \left(Q_a \frac{\partial G^a}{\partial u_{tx^i x^j x^k}^\alpha} \right) + \dots \right) \\
&\quad + D_j P^\alpha \left(Q_a \frac{\partial G^a}{\partial u_{x^i x^j}^\alpha} - D_k \left(Q_a \frac{\partial G^a}{\partial u_{x^i x^j x^k}^\alpha} \right) + D_k D_l \left(Q_a \frac{\partial G^a}{\partial u_{x^i x^j x^k x^l}^\alpha} \right) + \dots \right) \\
&\quad + \dots
\end{aligned} \tag{3.7}$$

modulo trivial terms $-D_t \Theta^i + D_j \Gamma^{ij}$. The multiplier of this conserved current is given by

$$\begin{aligned}
Q_a^{\hat{\mathbf{X}}} &= R_P^*(Q)_a - R_Q^*(P)_a \\
&= \left(R_P^{(0)b} Q_b - D_t (R_P^{(1)b} Q_b) - D_i (R_P^{(1)bi} Q_b) + D_t^2 (R_P^{(2)b} Q_b) \right. \\
&\quad \left. + D_t D_i (R_P^{(2)bi} Q_b) + D_i D_j (R_P^{(2)bij} Q_b) + \dots \right) \\
&\quad - \left(R_Q^{(0)a\alpha} P^\alpha - D_t (R_Q^{(1)a\alpha} P^\alpha) - D_i (R_Q^{(1)i} P^\alpha) + D_t^2 (R_Q^{(2)a\alpha} P^\alpha) \right. \\
&\quad \left. + D_t D_i (R_Q^{(2)i} P^\alpha) + D_i D_j (R_Q^{(2)ij} P^\alpha) + \dots \right).
\end{aligned} \tag{3.8}$$

Hence, the conserved currents $\Psi_G(P, Q)$ and $\Phi_{\hat{\mathbf{X}}}$ are trivial iff $Q_a^{\hat{\mathbf{X}}} = 0$ vanishes identically.

This result has a straightforward proof by comparing the multipliers for $\Psi_f(P, Q)$ and $\Phi_{\hat{\mathbf{X}}}$. Details for the case of a single PDE with two independent variables t, x are given in Ref.[12], while a full proof for the variational case is shown in Ref.[18]. The present general case is similar, and so a detailed proof will be omitted.

Theorem 3 can be used to provide a direct characterization for when a conservation law is invariant, or more generally homogenous, under the action of a symmetry, as defined in Ref.[12]. For a given infinitesimal symmetry (2.11), a conservation law (2.2) is *symmetry-invariant* iff the symmetry action on the corresponding conserved current $\Phi = (T, X^i)$ yields a trivial current,

$$\Phi_{\mathbf{X}}|_{\mathcal{E}} = (\text{pr} \hat{\mathbf{X}}(T)|_{\mathcal{E}}, \text{pr} \hat{\mathbf{X}}(X^i)|_{\mathcal{E}}) = (D_i \Theta^i, -D_t \Theta^i + D_j \Gamma^{ij}). \tag{3.9}$$

(This generalizes the notion introduced in Ref.[9] for singling out conserved currents that are strictly unchanged under the action of a symmetry.) A natural extension is to allow a conserved current to be equivalent to a multiple of itself under the symmetry action,

$$\Phi_{\mathbf{X}}|_{\varepsilon} - \lambda \Phi|_{\varepsilon} = ((\text{pr}\hat{\mathbf{X}}(T) - \lambda T)|_{\varepsilon}, (\text{pr}\hat{\mathbf{X}}(X^i) - \lambda X^i)|_{\varepsilon}) + (D_i\Theta^i, -D_t\Theta^i + D_j\Gamma^{ij}) \quad (3.10)$$

where λ is a non-zero constant. This corresponds to a conservation law (2.2) being *symmetry-homogeneous* under the action of an infinitesimal symmetry (2.9).

The condition of symmetry-invariance and symmetry-homogeneity for conservation laws has a simple formulation in terms of multipliers, which follows immediately from Theorem 3.

Theorem 4. *A conservation law (2.2) is homogeneous (3.10) under the action of an infinitesimal symmetry (2.9) iff its multiplier (2.25) satisfies the condition*

$$R_P^*(Q)_a - R_Q^*(P)_a = \lambda Q_a \quad (3.11)$$

for some constant λ . The conservation law is invariant (3.9) iff $\lambda = 0$. If a PDE system is an Euler-Lagrange system (2.33), then every conservation law is invariant under the variational symmetry corresponding to its multiplier.

Corollary 1. (i) Under the action of an infinitesimal symmetry (2.9), a conserved quantity (2.7) on a spatial domain $\Omega \subseteq \mathbb{R}^n$ is unchanged modulo an arbitrary boundary term, $\hat{\mathbf{X}}C[u] = \int_{\partial\Omega} \Theta^i dA_i$, iff the corresponding conservation law (2.2) is symmetry invariant. (ii) A conserved quantity (2.7) is mapped into itself (modulo an arbitrary boundary term) under a symmetry iff the corresponding conservation law (2.2) is symmetry homogeneous. In particular, under the action of a symmetry transformation $\exp(\epsilon\hat{\mathbf{X}})$ with parameter ϵ , a symmetry-homogeneous conserved quantity $C[u]$ is mapped into $\exp(\epsilon\lambda)C(u)$ whenever all boundary terms vanish.

One very useful application of these results is that the formula (3.6)–(3.7) can be used to construct the conserved current determined by a given multiplier in the important case when a PDE system admits a scaling symmetry. This was first developed and applied in Ref.[15]. Here we summarize the main result which will be used in the examples in the next section. (See Ref.[12, 15] for a proof.)

Proposition 2. *Suppose a normal PDE system (2.1) possesses a scaling symmetry*

$$t \rightarrow \lambda^p t, \quad x^i \rightarrow \lambda^{q^{(i)}} x^i, \quad u^\alpha \rightarrow \lambda^{r^{(\alpha)}} u^\alpha \quad (3.12)$$

where $p, q^{(i)}, r^{(\alpha)}$ are constants. Let Q_a be the multiplier for a conservation law in which the components of the conserved current $\Phi = (T, X^i)$ are scaling homogeneous, $(T, X^i) \rightarrow (\lambda^k T, \lambda^{k^{(i)}} X^i)$. Then, in terms of the characteristic functions $P^\alpha = r^{(\alpha)} u^\alpha - (p u_t^\alpha + q^{(i)} x^i u_{x^i}^\alpha)$ of the scaling symmetry, the conserved current $\Psi_G(P, Q)$ is equivalent to a multiple $w = k + \sum_{i=1}^n q^{(i)}$ of the conserved current Φ . This multiple w is equal to the scaling weight of the conserved quantity $C[u] = \int_\Omega T d^n x$. Hence, in the case when the conserved quantity is homogeneous under the scaling symmetry, so that $w \neq 0$, the components of the conserved current are given by $\Phi = (1/w)\Psi_G(P, Q)$ up to equivalence.

4. EXAMPLES

We will now consider several different examples of PDEs and PDE systems arising in a variety of physical applications, including water waves, fluid flow, and gas dynamics. For each example, we first show how to set up and apply the multiplier method to obtain all low-order conservation laws, and next we examine the symmetry properties of these conservation laws. All calculations have been carried out in Maple.

4.1. Nonlinear hyperbolic equation. Our first example is the nonlinear PDE

$$u_{tx} + u_x + ku^p = 0, \quad p > 1, k \neq 0. \quad (4.1)$$

This is a hyperbolic equation which arises, for $p = 2$, from gas dynamics when a generating function is formulated for the moments of the gas velocity distribution function exhibiting a Maxwellian tail [19]. We will refer to the PDE (4.1) as the generalized Maxwellian tails (gMT) equation. While the gMT equation is not an Euler-Lagrange equation as it stands, it does come from a Lagrangian $L = e^{2t}(\frac{1}{2}u_t u_x - \frac{k}{p+1}u^{p+1})$ through a variational integrating factor e^{2t} , as shown by $E_u(L) = -e^{2t}G$ where $G = u_{tx} + u_x + ku^p$. Note the gMT equation is normal, as it has the leading derivative u_{tx} .

The determining equation (2.12) for infinitesimal symmetries $\mathbf{X} = P\partial_u$ of the gMT equation (4.1) is given by

$$(D_t D_x P + D_x P + kpu^{p-1}P)|_{\mathcal{E}} = 0. \quad (4.2)$$

A straightforward computation of point and contact symmetries yields the characteristic functions

$$P_1 = -u_t, \quad P_2 = -u_x, \quad P_3 = e^{(p-1)t}(u_t + u), \quad P_4 = (p-1)xu_x + u, \quad (4.3)$$

from which we obtain

$$R_{P_1} = -D_t, \quad R_{P_2} = -D_x, \quad R_{P_3} = e^{(p-1)t}(D_t + p), \quad R_{P_4} = (p-1)x D_x + p \quad (4.4)$$

as given by equation (3.4). These symmetries consist of separate translations in t and x , a time-dependent dilation in u combined with a time-dependent shift in t , and a scaling in u and x , all of which are point symmetries [20]

$$\mathbf{X}_1 = \partial_t, \quad \mathbf{X}_2 = \partial_x, \quad \mathbf{X}_3 = e^{(p-1)t}(-\partial_t + u\partial_u), \quad \mathbf{X}_4 = (1-p)x\partial_x + u\partial_u, \quad (4.5)$$

with the respective group actions

$$t \rightarrow t + \epsilon, \quad (4.6)$$

$$x \rightarrow x + \epsilon, \quad (4.7)$$

$$t \rightarrow \frac{1}{1-p} \ln((p-1)\epsilon + e^{(1-p)t}), \quad u \rightarrow ((p-1)\epsilon + e^{(1-p)t})^{p-1}u \quad (4.8)$$

$$x \rightarrow e^{(p-1)\epsilon}x, \quad u \rightarrow e^\epsilon u \quad (4.9)$$

in terms of a parameter ϵ .

Since the gMT equation (4.1) has a Lagrangian formulation, all conservation laws of this equation arise from multipliers given by the relation

$$Q = e^{2t}P \quad (4.10)$$

where P is the characteristic of a variational symmetry $\hat{\mathbf{X}} = P\partial_u$ of the gMT equation. Variational contact symmetries and variational point symmetries thereby correspond to multipliers with the form

$$Q(t, x, u, u_t, u_x). \quad (4.11)$$

Note that u_t, u_x are the only derivatives of u that can be differentiated to obtain the leading derivative u_{tx} in the gMT equation, and hence multipliers having the form (4.11) generate all low-order conservation laws for the gMT equation. The determining system for these multipliers (4.10)–(4.11) consists of the adjoint-symmetry determining equation (2.28) which is given by

$$(D_t D_x Q - D_x Q + kpu^{p-1}Q)|_{\mathcal{E}} = 0, \quad (4.12)$$

and the Helmholtz equations (2.32) which can be shown to reduce to the equation

$$Q_u - Q_{u_t} = 0. \quad (4.13)$$

Hence, a symmetry of the gMT equation (4.1) is variational iff its characteristic satisfies the condition

$$P_u - P_{u_t} = 0. \quad (4.14)$$

From Theorem 2, this condition is equivalent to invariance of the Lagrangian, $\text{pr}\hat{\mathbf{X}}(L) = D_t A + D_x B$ for some differential functions $A(t, x, u, u_t, u_x)$ and $B(t, x, u, u_t, u_x)$.

When the variational symmetry condition (4.14) is applied to a linear combination of the four point symmetries (4.5), it shows that $\mathbf{X}_1 - \mathbf{X}_4$, \mathbf{X}_2 , \mathbf{X}_3 generate all variational point symmetries. These three variational symmetries correspond to the respective multipliers

$$Q_1 = -e^{2t}(u_t + (p-1)xu_x + u), \quad Q_2 = -e^{2t}u_x, \quad Q_3 = e^{(p+1)t}(u_t + u) \quad (4.15)$$

from which we obtain

$$R_{Q_1} = -e^{2t}((p-1)x D_x + D_t + p), \quad R_{Q_2} = -e^{2t}D_x, \quad R_{Q_3} = e^{(p+1)t}(D_t + p) \quad (4.16)$$

as given by equation (3.4).

Each multiplier (4.15) satisfies the characteristic equation

$$D_t T + D_x X = QG, \quad Q = T_{u_x} + X_{u_t}, \quad G = u_{tx} + u_x + ku^p \quad (4.17)$$

where the conserved current $\Phi = (T(t, x, u, u_t, u_x), X(t, x, u, u_t, u_x))$ can be obtained directly by either integration of the characteristic equation (4.17) or use of the scaling formula $\Psi_G((p-1)xu_x + u, Q)$ from Proposition 2. In particular, the components of the scaling formula are given by

$$\Psi_G^t = -((p-1)xu_x + u)D_x Q, \quad \Psi_G^x = (u_t - (p-1)ku^p + u)Q. \quad (4.18)$$

We obtain, modulo the addition of a trivial current,

$$T_1 = e^{2t}\left(\frac{p-1}{2}xu_x^2 + \frac{k}{p+1}u^{p+1}\right), \quad X_1 = e^{2t}\left(\frac{1}{2}(u + u_t)^2 + \frac{k(p-1)}{p+1}xu^{p+1}\right) \quad (4.19)$$

$$T_2 = \frac{1}{2}e^{2t}u_x^2, \quad X_2 = \frac{k}{p+1}e^{2t}u^{p+1} \quad (4.20)$$

$$T_3 = \frac{1}{p+1}e^{(p+1)t}u^{p+1}, \quad X_3 = \frac{1}{2}e^{(p+1)t}(u + u_t)^2 \quad (4.21)$$

These conserved currents represent conservation of three energy-momentum quantities $\int_{\Omega} T_i dx$ for the gMT equation (4.1) on any spatial domain $\Omega \subseteq \mathbb{R}$.

We will now study the symmetry properties of the conservation laws (4.19)–(4.21). Consider the vector space of conserved currents

$$T = a_1 T_1 + a_2 T_2 + a_3 T_3, \quad a_i = \text{const.}, \quad (4.22)$$

and the algebra of point symmetries

$$\mathbf{X} = c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 + c_3 \mathbf{X}_3 + c_4 \mathbf{X}_4, \quad c_j = \text{const.} \quad (4.23)$$

A conservation law $(D_t T + D_x X)|_{\mathcal{E}} = 0$ is homogeneous under the symmetry \mathbf{X} iff condition (3.11) is satisfied, where the characteristic function of the symmetry generator is given by $P = c_1 P_1 + c_2 P_2 + c_3 P_3 + c_4 P_4$, and the multiplier for the conservation law is given by $Q = a_1 Q_1 + a_2 Q_2 + a_3 Q_3$. By using equations (4.4) and (4.16), we find that the condition (3.11) splits with respect to t, x, u, u_t, u_x into a system of bilinear equations on c_j and a_i :

$$a_1(\lambda - 2c_1 - 2c_4) = 0 \quad (4.24a)$$

$$((p+1)c_4 + 2c_1 - \lambda)a_2 + (p-1)a_1 c_2 = 0 \quad (4.24b)$$

$$((p+1)c_1 + 2c_4 - \lambda)a_3 - (p-1)a_1 c_3 = 0 \quad (4.24c)$$

The solutions for c_j in terms of a_i determine the symmetry-homogeneity properties of the conserved current $\Phi = (T, X)$ modulo trivial currents $\Phi_{\text{triv}} = (D_x \Theta, -D_t \Theta)$. Note that any solution can be scaled by a constant factor, $c_j \rightarrow \gamma c_j$ with $\gamma \neq 0$. By considering the subspaces generated by $\{a_i\}$, solving the system (4.24) in each case, and merging the solutions, we obtain the conditions

$$a_1^2 + a_2^2 + a_3^2 \neq 0; \quad a_3 c_1 - a_1 c_3 = 0, \quad a_1 c_2 + a_2 c_4 = 0, \quad \lambda = 2(c_1 + c_4) \quad (4.25a)$$

$$a_2 \neq 0, \quad a_1 = a_3 = 0; \quad \lambda = (p+1)c_4 + 2c_1 \quad (4.25b)$$

$$a_3 \neq 0, \quad a_1 = a_2 = 0; \quad \lambda = (p+1)c_1 + 2c_4 \quad (4.25c)$$

Hence, we conclude the following.

(1) The symmetry properties of the vector space $a_1 \Phi_1 + a_2 \Phi_2 + a_3 \Phi_3 + \Phi_{\text{triv}}$ for arbitrary a_i consist of: (i) invariance under $a_1(\mathbf{X}_1 - \mathbf{X}_4) + a_2 \mathbf{X}_2 + a_3 \mathbf{X}_3$; (ii) homogeneity under $a_1 \mathbf{X}_1 + a_3 \mathbf{X}_3$ and $a_1 \mathbf{X}_4 - a_2 \mathbf{X}_2$, with $\lambda = 2a_1$.

(2) The only additional symmetry properties of the vector space $a_1 \Phi_1 + a_2 \Phi_2 + a_3 \Phi_3 + \Phi_{\text{triv}}$ consist of: (i) invariance of the subspace $a_2 \Phi_2 + \Phi_{\text{triv}}$ under \mathbf{X}_3 and $2\mathbf{X}_4 - (p+1)\mathbf{X}_1$; (ii) invariance of the subspace $a_3 \Phi_3 + \Phi_{\text{triv}}$ under \mathbf{X}_2 and $2\mathbf{X}_1 - (p+1)\mathbf{X}_4$; (iii) homogeneity of the subspace $a_2 \Phi_2 + \Phi_{\text{triv}}$ under \mathbf{X}_4 with $\lambda = p+1$, and \mathbf{X}_1 with $\lambda = 2$; (iv) homogeneity of the subspace $a_3 \Phi_3 + \Phi_{\text{triv}}$ under \mathbf{X}_1 with $\lambda = p+1$, and \mathbf{X}_4 with $\lambda = 2$.

From these properties, it is simple to work out the symmetries for which each energy-momentum quantity $\int_{\Omega} T_i dx$ is invariant (modulo an endpoint term $\Theta|_{\partial\Omega}$). In particular, $\int_{\Omega} T_1 dx$ is invariant under $\mathbf{X}_1 - \mathbf{X}_4$; $\int_{\Omega} T_2 dx$ is invariant under $\mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 - \frac{1}{2}(p+1)\mathbf{X}_1$; and $\int_{\Omega} T_3 dx$ is invariant under $\mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_1 - \frac{1}{2}(p+1)\mathbf{X}_4$. More generally, with boundary conditions such that all endpoint terms vanish, $\int_{\Omega} T_1 dx$ is mapped into $e^{2\epsilon} \int_{\Omega} T_1 dx$ under time-translations (4.6) and scalings (4.9); $\int_{\Omega} T_2 dx$ is mapped into $e^{2\epsilon} \int_{\Omega} T_2 dx$ under time-translations (4.6), and into $e^{(p+1)\epsilon} \int_{\Omega} T_2 dx$ under scalings (4.9); $\int_{\Omega} T_3 dx$ is mapped into $e^{(p+1)\epsilon} \int_{\Omega} T_3 dx$ under time-translations (4.6), and into $e^{2\epsilon} \int_{\Omega} T_3 dx$ under scalings (4.9).

It is interesting to note that every conserved quantity $\int_{\Omega} a_1 T_1 + a_2 T_2 + a_3 T_3 dx$ is invariant under the variational symmetry $\hat{\mathbf{X}} = e^{-2t} Q \partial_u$ that corresponds to its multiplier $Q = a_1 Q_1 + a_2 Q_2 + a_3 Q_3$ through the relation (4.10). This relation also shows that $R_Q = e^{2t} R_P$, from

which we see that the symmetry-invariance condition $R_P^*(Q) - R_Q^*(P) = 0$ holds identically when $Q = e^{2t}P$. Consequently, the symmetry-invariance property stated in Theorem 4 for conservation laws of Euler-Lagrange PDEs extends to the case of PDEs that acquire the form of Euler-Lagrange equations when a variational integrating factor is introduced.

As a final result, we observe that all three energy-momentum conservation laws (4.19)–(4.21) are homogeneous under the non-variational symmetries \mathbf{X}_1 and \mathbf{X}_4 , while under the variational symmetry $\mathbf{X}_1 - \mathbf{X}_4$, the two conservation laws (4.20) and (4.21) are homogeneous and the conservation law (4.19) is invariant. Thus, in contrast to Noethers theorem, the symmetry homogeneity condition for multipliers can yield a conservation law from a non-variational symmetry and also can yield more than one conservation law from a single variational symmetry.

4.2. Dispersive nonlinear wave equation. Our next example is the generalized Korteweg-de Vries (gKdV) equation

$$u_t + u_{xxx} + ku^p u_x = 0, \quad p > 0, k \neq 0. \quad (4.26)$$

This PDE is a dispersive nonlinear wave equation, which reduces to the KdV equation when $p = 1$ and the modified KdV equation when $p = 2$. If a potential v is introduced by $u = v_x$, then the Lagrangian $L = -\frac{1}{2}v_t v_x + \frac{1}{2}v_{xx}^2 + \frac{k}{(p+1)(p+2)}v_x^{p+2}$ yields $E_v(L) = v_{tx} + v_{xxx} + kv_x^p v_{xx} = G$, where $G = u_t + u_{xxx} + ku^p u_x = 0$. Note this is a normal PDE, as it has the leading derivative u_t (or u_{xxx}).

The determining equation (2.12) for infinitesimal symmetries $\mathbf{X} = P\partial_u$ of the gKdV equation (4.26) is given by

$$(D_t P + D_x^3 P + ku^p D_x P + kpu^{p-1}P)|_{\mathcal{E}} = 0. \quad (4.27)$$

It is well known that the gKdV equation has no contact symmetries and that all of its point symmetries are generated by a time translation, a space translation, and a scaling, when $p \neq 0$, plus a Galilean boost, when $p = 1$. For these symmetries

$$\mathbf{X}_1 = \partial_t, \quad \mathbf{X}_2 = \partial_x, \quad \mathbf{X}_3 = 3t\partial_t + x\partial_x - (2/p)u\partial_u, \quad (4.28)$$

$$\mathbf{X}_4 = kt\partial_x + \partial_u, \quad p = 1, \quad (4.29)$$

the characteristic functions are given by

$$\begin{aligned} P_1 &= -u_t, & P_2 &= -u_x, & P_3 &= -(3tu_t + xu_x + (2/p)u), \\ P_4 &= 1 - ktu_x, & p &= 1, \end{aligned} \quad (4.30)$$

from which we obtain

$$R_{P_1} = -D_t, \quad R_{P_2} = -D_x, \quad R_{P_3} = -(3tD_t + xD_x + (3 + 2/p)), \quad R_{P_4} = -ktD_x \quad (4.31)$$

as given by equation (3.4). Their respective group actions are given by

$$t \rightarrow t + \epsilon, \quad (4.32)$$

$$x \rightarrow x + \epsilon, \quad (4.33)$$

$$t \rightarrow e^{3\epsilon}t, \quad x \rightarrow e^\epsilon x, \quad u \rightarrow e^{-(2/p)\epsilon}u \quad (4.34)$$

$$x \rightarrow x + \epsilon kt, \quad u \rightarrow u + \epsilon \quad (4.35)$$

in terms of a parameter ϵ .

The characteristic equation for conservation laws of the gKdV equation (4.26) is given by

$$D_t T + D_x X = QG, \quad Q = E_u(T) \quad (4.36)$$

where T is a function of t, x, u , and x -derivatives of u , with all t -derivatives of u being eliminated from T through the gKdV equation $u_t = -(u_{xxx} + ku^p u_x)$. Thus, multipliers Q will be functions only of t, x, u , and x -derivatives of u . The determining system (2.28) and (2.32) for multipliers then consists of the adjoint-symmetry equation

$$-(D_t Q + D_x^3 Q + ku^p D_x Q)|_{\mathcal{E}} = 0 \quad (4.37)$$

and the Helmholtz equations

$$Q' - Q'^* = 0. \quad (4.38)$$

Since the gKdV equation (4.26) is an Euler-Lagrange equation in terms of the potential v , all multipliers are the same as characteristics of variational symmetries $\hat{\mathbf{X}} = Q\partial_v$ acting on v . When prolonged to $u = v_x$, these symmetries have the characteristic form

$$P = D_x Q \quad (4.39)$$

corresponding to $\hat{\mathbf{X}} = D_x Q \partial_u$, where P contains no t -derivatives of u .

When t -derivatives of u are eliminated, the characteristic functions of the point symmetries (4.28)–(4.29) of the gKdV equation (4.26) are given by

$$\begin{aligned} P_1 &= u_{xxx} + ku^p u_x, & P_2 &= -u_x, & P_3 &= 3tu_{xxx} + (3tku^p - x)u_x - (2/p)u, \\ P_4 &= 1 - ktu_x, & p &= 1, \end{aligned} \quad (4.40)$$

which are each of third order. It is now straightforward to derive the conditions for a point symmetry of the gKdV equation (4.26) to yield a conservation law multiplier. First, the correspondence relation (4.39) holds iff

$$0 = E_u(P) = P_u - D_x P_{u_x} + D_x^2 P_{u_{xx}} - D_x^3 P_{u_{xxx}}. \quad (4.41)$$

Second, from this relation (4.39), the symmetry determining equation (4.27) becomes $D_x(D_t Q + D_x^3 Q + ku^p D_x Q)|_{\mathcal{E}} = 0$ where $Q = D_x^{-1} P$ is of second order. This equation implies $(D_t Q + D_x^3 Q + ku^p D_x Q)|_{\mathcal{E}} = f(t)$, but since P is a homogeneous function (i.e. $P|_{u=0} = 0$), then Q can be assumed to be homogeneous, yielding $f(t) = 0$. Hence Q thereby satisfies the adjoint-symmetry equation (4.37). Next, since Q is of second order, the Helmholtz equations (4.38) reduce to the equation

$$Q_{u_x} - D_x Q_{u_{xx}} = 0. \quad (4.42)$$

Both this equation (4.42) and equation (4.39) now can be split with respect to all derivatives of u higher than second-order. This splitting yields $P_{u_{xxx}} = Q_{u_{xx}}$ and $P_{u_{xx}} = 2Q_{u_x}$, from which it can be shown that the Helmholtz equation (4.42) is equivalent to the condition

$$P_{u_{xx}} - 2D_x P_{u_{xxx}} = 0. \quad (4.43)$$

Therefore, the conditions (4.43) and (4.41) are necessary and sufficient for a point symmetry to yield a multiplier through the correspondence relation (4.39). The resulting multipliers

$$Q(t, x, u, u_x, u_{xx}) \quad (4.44)$$

will correspond to variational point symmetries $\hat{\mathbf{X}} = Q\partial_v$ for the gKdV equation expressed in Lagrangian form using the potential $u = v_x$. Each multiplier determines a conserved current

$$\Phi = (T(t, x, u, u_x), X(t, x, u, u_t, u_x, u_{xx})) \quad (4.45)$$

whose form arises from splitting the characteristic equation (4.36) with respect to u_{tx} , u_{xxx} , u_{xxxx} . Moreover, since u_x, u_{xx} are the only derivatives of u that can be differentiated to yield a leading derivative of the gKdV equation (4.26), multipliers with the form (4.44) generate all low-order conserved currents (4.45) for the gKdV equation.

We now apply conditions (4.43) and (4.41) to a linear combination of the point symmetry characteristic functions (4.40). This yields the multipliers

$$Q_1 = u_{xx} + \frac{k}{p+1}u^{p+1}, \quad Q_2 = -u, \quad (4.46)$$

$$Q_3 = 3t(u_{xx} + ku^3) - xu, \quad p = 2, \quad (4.47)$$

$$Q_4 = x - ktu, \quad p = 1, \quad (4.48)$$

which can be seen to correspond to the point symmetries \mathbf{X}_1 , \mathbf{X}_2 , \mathbf{X}_3 with $p = 2$, and \mathbf{X}_4 . These multipliers are known to comprise all solutions of the determining system (4.37) and (4.38) for multipliers of at most second order [2], apart from the obvious constant solution

$$Q_5 = 1 \quad (4.49)$$

(which can be viewed as arising from a trivial symmetry, $P = 0$, by the correspondence relation (4.39)). Note we have

$$R_{Q_5} = 0, \quad R_{Q_1} = -D_x^2 - ku^p, \quad R_{Q_2} = 1, \quad (4.50)$$

$$R_{Q_3} = -3t(D_x^2 + ku^2) + x, \quad p = 2, \quad (4.51)$$

$$R_{Q_4} = kt, \quad p = 1, \quad (4.52)$$

as given by equation (3.4).

For each multiplier (4.46)–(4.49), a conserved current (4.45) can be obtained directly by either integration of the characteristic equation (4.36) or use of the scaling formula $\Psi_G(3tu_t + xu_x + (2/p)u, Q)$ from Proposition 2. In particular, the components of the scaling formula are given by

$$\begin{aligned} \Psi_G^t &= \left(\frac{2}{p}u + xu_x - 3t(u_{xxx} + ku^p u_x)\right)Q, \\ \Psi_G^x &= \left(\frac{2}{p}u + xu_x - 3t(u_{xxx} + ku^p u_x)\right)(ku^p Q + D_x^2 Q) \\ &\quad - \left((1 + \frac{2}{p})u_x + xu_{xx} - 3t(u_{xxxx} + k(u^p u_x)_x)\right)D_x Q \\ &\quad + \left(2(1 + \frac{1}{p})u_{xx} + xu_{xxx} - 3t(u_{xxxx} + k(u^p u_x)_{xx})\right)Q. \end{aligned} \quad (4.53)$$

We obtain, modulo the addition of a trivial current,

$$T_5 = u, \quad X_5 = u_{xx} + \frac{k}{p+1}u^{p+1} \quad (4.54)$$

$$T_1 = -\frac{1}{2}u_x^2 + \frac{k}{(p+1)(p+2)}u^{p+2}, \quad X_1 = \frac{1}{2}\left(u_{xx} + \frac{k}{p+1}u^{p+1}\right)^2 + u_t u_x \quad (4.55)$$

$$T_2 = -\frac{1}{2}u^2, \quad X_2 = -uu_{xx} + \frac{1}{2}u_x^2 - \frac{k}{p+2}u^{p+2} \quad (4.56)$$

$$T_3 = -t\left(\frac{3}{2}u_x^2 + \frac{k}{4}u^4\right) - \frac{1}{2}xu^2, \quad (4.57)$$

$$X_3 = \frac{3}{2}t\left(u_{xx} + \frac{k}{3}u^3\right)^2 - x\left(uu_{xx} - \frac{1}{2}u_x^2 + \frac{k}{4}u^4\right) + (3tu_t + u)u_x, \quad p = 2$$

$$T_4 = -\frac{1}{2}ktu^2 - xu, \quad (4.58)$$

$$X_4 = -kt\left(uu_{xx} - \frac{1}{2}u_x^2 - \frac{k}{3}u^3\right) + x\left(u_{xx} + \frac{k}{2}u^2\right) - u_x, \quad p = 1$$

These five conserved currents respectively represent conservation of mass, momentum, energy, Galilean energy, and Galilean momentum for the gKdV equation (4.26).

We will now study the symmetry properties of the conservation laws (4.54)–(4.58). Consider the vector space of conserved currents

$$\begin{aligned} T &= a_1 T_1 + a_2 T_2 + a_3 T_3 + a_4 T_4 + a_5 T_5, \\ a_i &= \text{const.}, \quad a_3 = 0 \text{ if } p \neq 2, \quad a_4 = 0 \text{ if } p \neq 1, \end{aligned} \quad (4.59)$$

and the algebra of point symmetries

$$\mathbf{X} = c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 + c_3 \mathbf{X}_3 + c_4 \mathbf{X}_4, \quad c_j = \text{const.}, \quad c_4 = 0 \text{ if } p \neq 1. \quad (4.60)$$

A conservation law $(D_t T + D_x X)|_{\mathcal{E}} = 0$ is homogeneous under the symmetry \mathbf{X} iff condition (3.11) is satisfied, where the characteristic of the symmetry generator is given by $P = c_1 P_1 + c_2 P_2 + c_3 P_3 + c_4 P_4$, and the multiplier for the conservation law is given by $Q = a_1 Q_1 + a_2 Q_2 + a_3 Q_3 + a_4 Q_4 + a_5 Q_5$. By using equations (4.31) and (4.50)–(4.52), we find that the condition (3.11) splits with respect to t, x, u, u_x, u_{xx} into a system of bilinear equations on c_j and a_i :

$$p \neq 1, 2: \quad a_2(p\lambda + (4-p)c_3) = 0 \quad (4.61a)$$

$$p \neq 1: \quad a_5(p\lambda + (2-p)c_3) = 0 \quad (4.61b)$$

$$p \neq 2: \quad a_1(p\lambda + (p+4)c_3) = 0 \quad (4.61c)$$

$$p = 1: \quad a_4\lambda = 0, \quad k(a_1c_4 - a_4c_1) + a_2(\lambda + 3c_3) = 0, \quad a_2c_4 - a_4c_2 + a_5(\lambda + c_3) = 0 \quad (4.61d)$$

$$p = 2: \quad a_3\lambda = 0, \quad 3a_3c_1 - a_1(\lambda + 3c_3) = 0, \quad a_3c_2 - a_2(\lambda + c_3) = 0 \quad (4.61e)$$

The solutions for c_j in terms of a_i determine the symmetry-homogeneity properties of the conserved current $\Phi = (T, X)$ modulo trivial currents $\Phi_{\text{triv}} = (D_x \Theta, -D_t \Theta)$. By considering the subspaces generated by $\{a_i\}$, solving the system (4.61) in each case, and merging the solutions, we get the conditions

$$p \neq 1: \quad a_5 \neq 0, \quad a_1 = a_2 = a_4 = 0; \quad \lambda = (1 - 2/p)c_3 \quad (4.62a)$$

$$p \neq 1: \quad a_2 \neq 0, \quad a_1 = a_5 = a_3 = 0; \quad \lambda = (1 - 4/p)c_3 \quad (4.62b)$$

$$p \neq 1: \quad a_1 \neq 0, \quad a_2 = a_5 = a_3 = 0; \quad \lambda = -(1 + 4/p)c_3 \quad (4.62c)$$

$$p = 1: \quad a_1^2 + a_2^2 + a_5^2 \neq 0, \quad a_2^2 = 2ka_1a_5, \quad a_4 = 0; \quad a_2c_4 = 4a_5c_3, \quad \lambda = -5c_3 \quad (4.62d)$$

$$p = 1: \quad a_2^2 + a_5^2 \neq 0, \quad a_1 = a_4 = 0; \quad a_2c_4 = 2a_5c_3, \quad \lambda = -3c_3 \quad (4.62e)$$

$$p = 1: \quad a_5 \neq 0, \quad a_1 = a_2 = a_4 = 0; \quad \lambda = -c_3 \quad (4.62f)$$

$$p = 1: \quad a_2 \neq 0, \quad a_1 = a_5 = a_4 = 0; \quad c_4 = 0, \quad \lambda = -3c_3 \quad (4.62g)$$

$$p = 1: \quad a_1 \neq 0, \quad a_2 = a_5 = a_4 = 0; \quad c_4 = 0, \quad \lambda = -5c_3 \quad (4.62h)$$

$$p = 1: \quad a_1^2 + a_2^2 + a_4^2 + a_5^2 \neq 0; \quad c_3 = 0, \quad a_4c_1 = a_1c_4, \quad a_4c_2 = a_2c_4, \quad \lambda = 0 \quad (4.62i)$$

$$p = 1: \quad a_2^2 + a_4^2 + a_5^2 \neq 0, \quad a_1 = 0; \quad ka_4c_1 = 3a_2c_3, \quad a_4c_2 = a_2c_4 + a_5c_3, \quad \lambda = 0 \quad (4.62j)$$

$$p = 1: \quad a_1^2 + a_5^2 + a_4^2 \neq 0, \quad a_2 = 0; \quad c_2 = c_3 = 0, \quad a_4c_1 = a_1c_4, \quad \lambda = 0 \quad (4.62k)$$

$$p = 2: \quad a_1^2 + a_2^2 + a_5^2 + a_3^2 \neq 0; \quad a_3c_1 = a_1c_3, \quad a_3c_2 = a_2c_3, \quad a_1c_2 = a_2c_1, \quad \lambda = 0 \quad (4.62l)$$

Hence, we conclude the following.

(1) For $p \neq 1, 2$, the symmetry properties of the vector space $a_1\Phi_1 + a_2\Phi_2 + a_5\Phi_5 + \Phi_{\text{triv}}$ are generated by: (i) invariance under $\mathbf{X}_1, \mathbf{X}_2$; (ii) homogeneity of the subspace $a_1\Phi_1 + \Phi_{\text{triv}}$ under \mathbf{X}_3 with $\lambda = -1 - 4/p$; (iii) homogeneity of the subspace $a_2\Phi_2 + \Phi_{\text{triv}}$ under \mathbf{X}_3 with $\lambda = 1 - 4/p$; (iv) homogeneity of the subspace $a_5\Phi_5 + \Phi_{\text{triv}}$ under \mathbf{X}_3 with $\lambda = 1 - 2/p$.

(2) For $p = 1$, the symmetry properties of the vector space $a_1\Phi_1 + a_2\Phi_2 + a_5\Phi_5 + a_4\Phi_4 + \Phi_{\text{triv}}$ are generated by: (i) invariance of the subspace $a_1\Phi_1 + a_2\Phi_2 + a_4\Phi_4 + a_5\Phi_5 + \Phi_{\text{triv}}$ under $a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + a_4\mathbf{X}_4$; (ii) invariance of the subspace $a_2\Phi_2 + a_5\Phi_5 + a_4\Phi_4 + \Phi_{\text{triv}}$ under $a_2\mathbf{X}_2 + a_4\mathbf{X}_4$ and $\frac{3}{k}a_2\mathbf{X}_1 + a_5\mathbf{X}_2 + a_4\mathbf{X}_3$; (iii) invariance of the subspace $a_1\Phi_1 + a_5\Phi_5 + a_4\Phi_4 + \Phi_{\text{triv}}$ under $a_1\mathbf{X}_1 + a_4\mathbf{X}_4$; (iv) invariance of the subspace $a_1\Phi_1 + \Phi_{\text{triv}}$ under \mathbf{X}_2 ; (v) invariance of the subspace $a_5\Phi_5 + \Phi_{\text{triv}}$ under \mathbf{X}_1 ; (vi) homogeneity of the projective subspace $a_1\Phi_1 + a_2\Phi_2 + a_5\Phi_5 + \Phi_{\text{triv}}$, $a_2^2 = 2ka_1a_5$, under $a_2\mathbf{X}_3 + 4a_5\mathbf{X}_4$ with $\lambda = -5a_2$; (vii) homogeneity of the subspace $a_2\Phi_2 + a_5\Phi_5 + \Phi_{\text{triv}}$ under $a_2\mathbf{X}_3 + 2a_5\mathbf{X}_4$ with $\lambda = -3a_2$; (viii) homogeneity of the subspace $a_1\Phi_1 + \Phi_{\text{triv}}$ under \mathbf{X}_3 with $\lambda = -5$; (ix) homogeneity of the subspace $a_2\Phi_2 + \Phi_{\text{triv}}$ under \mathbf{X}_3 with $\lambda = -3$; (x) homogeneity of the subspace $a_5\Phi_5 + \Phi_{\text{triv}}$ under \mathbf{X}_3 with $\lambda = -1$.

(3) For $p = 2$, the symmetry properties of the vector space $a_1\Phi_1 + a_2\Phi_2 + a_5\Phi_5 + a_3\Phi_3 + \Phi_{\text{triv}}$ are generated by: (i) invariance under $a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + a_3\mathbf{X}_3$; (ii) invariance of the subspace $a_1\Phi_1 + \Phi_{\text{triv}}$ under \mathbf{X}_2 ; (iii) invariance of the subspace $a_2\Phi_2 + \Phi_{\text{triv}}$ under \mathbf{X}_1 ; (iv) invariance of the subspace $a_5\Phi_5 + \Phi_{\text{triv}}$ under $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$; (v) homogeneity of the subspace $a_1\Phi_1 + \Phi_{\text{triv}}$ under \mathbf{X}_3 with $\lambda = -3$; (vi) homogeneity of the subspace $a_2\Phi_2 + \Phi_{\text{triv}}$ under \mathbf{X}_3 with $\lambda = -1$.

From these properties, it follows that the mass, momentum, energy conservation laws (4.54)–(4.56) admitted for $p \neq 1, 2$ are homogeneous under the scaling symmetry \mathbf{X}_3 and invariant under the translation symmetries \mathbf{X}_1 and \mathbf{X}_2 , while the Galilean energy conservation law (4.57) admitted only for $p = 2$ is invariant under \mathbf{X}_3 and the Galilean momentum conservation law (4.58) admitted only for $p = 1$ is invariant under both \mathbf{X}_3 and \mathbf{X}_4 . In particular, the scaling symmetry \mathbf{X}_3 maps $\int_{\Omega} T_5 dx$ into $e^{(1-2/p)\epsilon} \int_{\Omega} T_5 dx$, $\int_{\Omega} T_1 dx$ into $e^{-(1+4/p)\epsilon} \int_{\Omega} T_1 dx$, and $\int_{\Omega} T_2 dx$ into $e^{(1-4/p)\epsilon} \int_{\Omega} T_2 dx$, if boundary conditions are imposed such that all endpoint terms vanish.

Additionally, for $p = 1$, a combined scaling and Galilean boost symmetry $a_2\mathbf{X}_3 + 4a_5\mathbf{X}_4$ maps the conserved quantity $\int_{\Omega} a_1T_1 \pm \sqrt{2ka_1a_5}T_2 + a_5T_5 dx$ into $e^{-5a_2\epsilon} \int_{\Omega} a_1T_1 \pm \sqrt{2ka_1a_5}T_2 + a_5T_5 dx$, and a similar symmetry $a_2\mathbf{X}_3 + 2a_5\mathbf{X}_4$ maps the conserved quantity $\int_{\Omega} a_2T_2 + a_5T_5 dx$ into $e^{-3a_2\epsilon} \int_{\Omega} a_2T_2 + a_5T_5 dx$, under suitable boundary conditions. These symmetry-homogeneous conserved quantities represent linear combinations of mass, momentum, and energy. Likewise, the conserved quantities $\int_{\Omega} a_2T_2 + a_4T_4 + a_1T_1 dx$ and $\int_{\Omega} a_2T_2 + a_4T_4 + a_5T_5 dx$ representing linear combinations of momentum, Galilean momentum, and energy or mass are invariant under the respective symmetries $a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + a_4\mathbf{X}_4$ and $a_2\mathbf{X}_1 + \frac{k}{3}(a_5\mathbf{X}_2 + a_4\mathbf{X}_3)$ representing t, x -translation symmetries combined with a Galilean boost or a scaling.

It is interesting to note that every conserved quantity $\int_{\Omega} a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4 dx$ is invariant under the variational symmetry $\hat{\mathbf{X}} = D_x Q \partial_u$ that corresponds to its multiplier $Q = a_1Q_1 + a_2Q_2 + a_3Q_3 + a_4Q_4$ through the relation (4.39). This relation also shows that $R_P = -D_x R_Q$, from which we see that the symmetry-invariance condition $R_P^*(Q) - R_Q^*(P) = 0$ holds identically when $D_x Q = P$. As a consequence, the symmetry-invariance property stated in Theorem 4 for conservation laws of Euler-Lagrange PDEs extends to the case of PDEs that have the form of Euler-Lagrange equations when a potential is introduced.

It is also interesting that, for the Lagrangian formulation of the gKdV equation, homogeneity under the non-variational scaling symmetry (4.34) when $p \neq 1, 2$ determines three conservation laws, in contrast to Noethers theorem which cannot yield a conservation law from this symmetry when $p \neq 1, 2$.

Finally, we remark that the correspondence relation $D_x Q = P$ between multipliers and variational symmetries can also be derived in a general way from the well-known Hamiltonian formulation of the gKdV equation (4.26), $u_t = D_x(\delta H/\delta u)$ where $H = -\int_{\mathbb{R}} T_1 dx$ is the Hamiltonian and D_x is a Hamiltonian operator. In particular, a general result in the theory of Hamiltonian PDEs [4] shows that every conserved density T admitted by a PDE of the form $u_t = D_x(\delta H/\delta u)$ yields a corresponding Hamiltonian symmetry whose characteristic function is simply $P = D_x Q$ with $Q = E_u(T)$.

We also mention that the bi-Hamiltonian structures [4] of the KdV equation and the modified KdV equation can be used to show that all of the higher-order conservation laws for these integrable equations are invariant under all of the higher-order symmetries generated from $\hat{\mathbf{X}} = u_x \partial_u$ by the recursion operator for each equation.

4.3. Nonlinear viscous fluid equation. Our third example is the PDE

$$u_t + uu_x = k(u^p u_x)_x, \quad p \neq 0, k \neq 0 \quad (4.63)$$

which reduces to Burgers' equation when $p = 1$. This nonlinear PDE (4.63) models a non-Newtonian viscous fluid where, for $p \neq 1$, the viscosity coefficients can depend nonlinearly on the fluid velocity [21]. Note that u_{xx} and u_t are leading derivatives, so the PDE is normal. We will refer to it as a generalized non-Newtonian Burgers' (gnNB) equation.

The determining equation (2.12) for infinitesimal symmetries $\mathbf{X} = P \partial_u$ of the gnNB equation (4.63) is given by

$$(D_t P + D_x(uP) - k D_x^2(u^p P))|_{\mathcal{E}} = 0. \quad (4.64)$$

Using the leading derivative $u_{xx} = \frac{1}{k} u^{-p}(u_t + uu_x) - p u^{-1} u_x^2$ for the gnNB equation, a straightforward computation of point and contact symmetries yields the characteristic functions

$$P_1 = -u_t, \quad P_2 = -u_x, \quad P_3 = -(p-2)tu_t - (p-1)xu_x + u, \quad (4.65)$$

from which we obtain

$$R_{P_1} = -D_t, \quad R_{P_2} = -D_x, \quad R_{P_3} = (2-p)tD_t + (1-p)xD_x + (3-p) \quad (4.66)$$

as given by equation (3.4). These symmetries are point symmetries, which consist of separate translations in t and x , and a scaling in t, x, u :

$$\mathbf{X}_1 = \partial_t, \quad \mathbf{X}_2 = \partial_x, \quad \mathbf{X}_3 = (p-2)t\partial_t + (p-1)x\partial_x + u\partial_u. \quad (4.67)$$

Their respective group actions consist of

$$t \rightarrow t + \epsilon, \quad (4.68)$$

$$x \rightarrow x + \epsilon, \quad (4.69)$$

$$t \rightarrow e^{(p-2)\epsilon}t, \quad x \rightarrow e^{(p-1)\epsilon}x, \quad u \rightarrow e^\epsilon u \quad (4.70)$$

in terms of a parameter ϵ .

The characteristic equation for conservation laws of the gnNB equation (4.63) is given by

$$D_t T + D_x X = QG, \quad Q = -\frac{1}{k} u^{-p} E_{u_x}(X), \quad G = u_t + uu_x - k(u^p u_x)_x \quad (4.71)$$

where X is a function of t, x, u, u_x , and t -derivatives of u, u_x , with all x -derivatives of u_x being eliminated from X through the gnNB equation, $u_{xx} = \frac{1}{k}u^{-p}(u_t + uu_x) - pu^{-1}u_x^2$. Consequently, multipliers Q are also functions only of t, x, u, u_x , and t -derivatives of u, u_x . Low-order conservation laws are defined by multipliers with the first-order form

$$Q(t, x, u, u_x) \quad (4.72)$$

where u_t is excluded because it cannot be differentiated to obtain a leading derivative u_t or u_{xx} of the gnNB equation. The corresponding conserved currents will have the form

$$\Phi = (T(t, x, u), X(t, x, u, u_x)) \quad (4.73)$$

which can be readily derived by splitting equation (4.71) with respect to u_t and its differential consequences.

The determining system (2.28) and (2.32) for all low-order conservation laws (4.73) consists of only the adjoint-symmetry equation

$$-(D_t Q + u D_x Q + k u^p D_x^2 Q)|_{\mathcal{E}} = 0 \quad (4.74)$$

since the Helmholtz equations (2.32) can be shown to hold identically due to Q not containing any x -derivatives of u_x . Hence, low-order adjoint-symmetries of the gnNB equation (4.63) are the same as low-order multipliers. There is no correspondence between multipliers and symmetries for this equation.

A straightforward computation of adjoint-symmetries (4.72) yields the multipliers

$$Q_1 = 1, \quad (4.75)$$

$$Q_2 = e^{-x/k}, \quad p = 1 \quad (4.76)$$

with

$$R_{Q_1} = 0, \quad R_{Q_2} = 0 \quad (4.77)$$

given by equation (2.29). For each multiplier (4.75)–(4.76), a conserved current (4.73) can be obtained directly by either integration of the characteristic equation (4.71) or use of the scaling formula $\Psi_G(u - (p-2)tu_t - (p-1)xu_x, Q)$ from Proposition 2. In particular, the components of the scaling formula are given by

$$\begin{aligned} \Psi_G^t &= (u - (p-2)tu_t - (p-1)xu_x)Q, \\ \Psi_G^x &= u\Psi_G^t - (u^{p+1} - (p-2)tu^p u_t - (p-1)xu^p u_x)D_x Q \\ &\quad + (2u^p u_x - (p-2)t(u^p u_t)_x - (p-1)x(u^p u_x)_x)Q. \end{aligned} \quad (4.78)$$

We obtain, modulo the addition of a trivial current,

$$T_1 = u, \quad X_1 = \frac{1}{2}u^2 - ku^p u_{xx} \quad (4.79)$$

$$T_2 = e^{-x/k}u, \quad X_2 = -ke^{-x/k}uu_x, \quad p = 1 \quad (4.80)$$

These two conserved currents respectively represent conservation of momentum and exponentially-weighted momentum for the gnNB equation (4.63).

We will now study the symmetry properties of the conservation laws (4.79)–(4.80). Consider the vector space of conserved currents

$$T = a_1 T_1 + a_2 T_2, \quad a_i = \text{const.}, \quad a_2 = 0 \text{ if } p \neq 1, \quad (4.81)$$

and the algebra of point symmetries

$$\mathbf{X} = c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 + c_3 \mathbf{X}_3, \quad c_j = \text{const.} \quad (4.82)$$

A conservation law $(D_t T + D_x X)|_{\mathcal{E}} = 0$ is homogeneous under the symmetry \mathbf{X} iff condition (3.11) is satisfied, where the characteristic of the symmetry generator is given by $P = c_1 P_1 + c_2 P_2 + c_3 P_3$, and the multiplier for the conservation law is given by $Q = a_1 Q_1 + a_2 Q_2$. By using equations (4.66) and (4.77), we find that the condition (3.11) splits with respect to $e^{-x/k}$ into a system of bilinear equations on c_j and a_i :

$$p \neq 1 : \quad pc_3 - \lambda = 0 \quad (4.83a)$$

$$p = 1 : \quad a_1(c_3 - \lambda) = 0, \quad a_2(c_2 + k(\lambda - c_3)) = 0 \quad (4.83b)$$

The solutions for c_j in terms of a_i determine the symmetry-homogeneity properties of the conserved current $\Phi = (T, X)$ modulo trivial currents $\Phi_{\text{triv}} = (D_x \Theta, -D_t \Theta)$. Solving the system (4.83), we get the conditions

$$p \neq 1 : \quad \lambda = pc_3 \quad (4.84a)$$

$$p = 1 : \quad a_1^2 + a_2^2 \neq 0, \quad c_2 = 0, \quad \lambda = c_3 \quad (4.84b)$$

$$p = 1 : \quad a_1^2 \neq 0, \quad a_2 = 0; \quad \lambda = c_3 \quad (4.84c)$$

$$p = 1 : \quad a_2^2 \neq 0, \quad a_1 = 0; \quad \lambda = c_3 - \frac{1}{k}c_2 \quad (4.84d)$$

Hence, we conclude the following.

(1) For $p \neq 1$, the symmetry properties of the vector space $a_1 \Phi_1 + \Phi_{\text{triv}}$ consist of: (i) invariance under $\mathbf{X}_1, \mathbf{X}_2$; (ii) homogeneity under \mathbf{X}_3 with $\lambda = p$.

(2) For $p = 1$, the symmetry properties of the vector space $a_1 \Phi_1 + a_2 \Phi_2 + \Phi_{\text{triv}}$ for arbitrary a_i consist of: (i) invariance under \mathbf{X}_1 ; (ii) homogeneity under X_3 with $\lambda = 1$.

(3) For $p = 1$, the only additional symmetry properties of the vector space $a_1 \Phi_1 + a_2 \Phi_2 + \Phi_{\text{triv}}$ consist of: (i) invariance of the subspace $a_1 \Phi_1 + \Phi_{\text{triv}}$ under X_2 ; (ii) homogeneity of the subspace $a_2 \Phi_2 + \Phi_{\text{triv}}$ under X_2 with $\lambda = -\frac{1}{k}$.

From these properties, it follows that the momentum conservation law (4.79) admitted for any p is homogeneous under the scaling symmetry \mathbf{X}_3 and invariant under the translation symmetries \mathbf{X}_1 and \mathbf{X}_2 , while the exponentially-weighted momentum conservation law (4.80) admitted only for $p = 1$ is homogeneous under both the scaling symmetry \mathbf{X}_3 and the x -translation symmetry \mathbf{X}_2 , and is invariant under the t -translation symmetry \mathbf{X}_1 as well as the combined symmetry $\mathbf{X}_3 + k\mathbf{X}_2$. In particular, the scaling symmetry \mathbf{X}_3 maps $\int_{\Omega} T_1 dx$ into $e^{p\epsilon} \int_{\Omega} T_1 dx$, and $\int_{\Omega} T_2 dx$ into $e^{\epsilon} \int_{\Omega} T_2 dx$, if boundary conditions are imposed such that all endpoint terms vanish.

4.4. Peakon equation. Our fourth example is the b -family peakon equation

$$m_t + um_x + bu_x m, \quad m = u - u_{xx}, \quad b \neq -1 \quad (4.85)$$

which arises [22] from the theory of shallow water waves and includes the Camassa-Holm equation [23] when $b = 2$ and the Degasperis-Procesi equation [24] when $b = 3$. As shown in Ref.[25], the b -family equation possesses multi-peakon solutions and has a Hamiltonian structure

$$m_t = \mathcal{D}(\delta H / \delta m), \quad H = \begin{cases} (b-1)^{-1} \int_{\mathbb{R}} m \, dx, & b \neq 1 \\ \int_{\mathbb{R}} m \ln(m) \, dx, & b = 1 \end{cases} \quad (4.86)$$

given in terms of the Hamiltonian operator

$$\mathcal{D} = (m^{1-1/b} D_x m^{1/b}) (D_x \Delta)^{-1} (m^{1/b} D_x m^{1-1/b}). \quad (4.87)$$

For studying conservation law and symmetries, it is more useful to work with the equivalent PDE

$$u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \quad b \neq -1. \quad (4.88)$$

Note that u_{xxx} and u_{txx} are leading derivatives in this b -family equation, so it is a normal PDE.

The determining equation (2.12) for infinitesimal symmetries $\mathbf{X} = P\partial_u$ of the b -family equation (4.88) is given by

$$(D_t \Delta P + u D_x \Delta P + bu_x \Delta P + b(u - u_{xx}) D_x P + (u_x - u_{xxx}) P)|_{\mathcal{E}} = 0 \quad (4.89)$$

with $\Delta = 1 - D_x^2$. A direct calculation shows that the b -family equation has no contact symmetries and that all of its point symmetries are generated by separate translations in t and x , a scaling in t, u , and, when $b = 0$, a Galilean boost. For these symmetries

$$\mathbf{X}_1 = \partial_t, \quad \mathbf{X}_2 = \partial_x, \quad \mathbf{X}_3 = u\partial_u - t\partial_t, \quad (4.90)$$

$$\mathbf{X}_4 = \partial_u + t\partial_x, \quad b = 0, \quad (4.91)$$

the characteristic functions are given by

$$P_1 = -u_t, \quad P_2 = -u_x, \quad P_3 = u + tu_t, \quad (4.92)$$

$$P_4 = 1 - tu_x, \quad b = 0, \quad (4.93)$$

from which we obtain

$$R_{P_1} = -D_t, \quad R_{P_2} = -D_x, \quad R_{P_3} = tD_t + 2, \quad R_{P_4} = -tD_x, \quad (4.94)$$

as given by equation (3.4). Their respective group actions are given by

$$t \rightarrow t + \epsilon, \quad (4.95)$$

$$x \rightarrow x + \epsilon, \quad (4.96)$$

$$t \rightarrow e^{-p\epsilon} t, \quad u \rightarrow e^\epsilon u \quad (4.97)$$

$$x \rightarrow x + \epsilon t, \quad u \rightarrow u + \epsilon \quad (4.98)$$

in terms of a parameter ϵ . (A symmetry classification presented in Ref.[26] is missing the case (4.98).)

To formulate the characteristic equation for conservation laws of the b -family equation (4.88), it is simplest to use the solved form $u_{xxx} = u_x + u^{-1}(bu_x(u - u_{xx}) - u_t + u_{txx})$ in terms of the leading derivative u_{xxx} . This yields

$$D_t T + D_x X = QG, \quad Q = -u^{-1} E_{u_{xx}}(X), \quad G = u_t - u_{txx} + (b+1)uu_x - bu_x u_{xx} - uu_{xxx} \quad (4.99)$$

where X is a function of t, x, u, u_x, u_{xx} , and t -derivatives of u, u_x, u_{xx} , with all x -derivatives of u_{xx} being eliminated from X . Consequently, multipliers Q are also functions only of t, x, u, u_x, u_{xx} , and t -derivatives of u, u_x, u_{xx} . Low-order conservation laws are defined by multipliers with the second-order form

$$Q(t, x, u, u_t, u_x, u_{tx}, u_{xx}) \quad (4.100)$$

where u_{tt} is excluded because it cannot be differentiated to obtain a leading derivative u_{xxx} or u_{txx} of the b -family equation. The corresponding conserved currents will have the form

$$\Phi = (T(t, x, u, u_x, u_{xx}), X(t, x, u, u_t, u_x, u_{tx}, u_{xx})) \quad (4.101)$$

which can be readily derived by splitting equation (4.99) with respect to u_{tt} and its differential consequences.

The determining system (2.28) and (2.32) for all low-order conservation laws (4.101) consists of only the adjoint-symmetry equation

$$(-D_t \Delta Q - D_x \Delta(uQ) + b \Delta(u_x Q) - b D_x((u - u_{xx})Q) + (u_x - u_{xxx})Q)|_{\mathcal{E}} = 0 \quad (4.102)$$

since the Helmholtz equations (2.32) can be shown to hold identically due to Q not containing any t -derivatives of u_{xx} . Hence, low-order adjoint-symmetries of the b -family equation (4.88) are the same as low-order multipliers. There is no correspondence between multipliers $Q(t, x, u, u_x, u_{xx}, u_t, u_{tx}, u_{txx}, \dots)$ and symmetry characteristics $P(t, x, u, u_x, u_{xx}, u_t, u_{tx}, u_{txx}, \dots)$ for this equation. (However, if the class of multipliers and symmetries is enlarged by including certain nonlocal variables that arise from the Hamiltonian structure, then a correspondence within this larger class does exist.)

A straightforward computation of adjoint-symmetries (4.100) yields the multipliers

$$Q_1 = (u - u_{xx})^{-1+1/b}, \quad b \neq 0 \quad (4.103)$$

$$Q_2 = 1 \quad (4.104)$$

$$Q_3 = u, \quad b = 2 \quad (4.105)$$

$$Q_4 = u(u - u_{xx}) + \frac{1}{2}(u^2 - u_x^2) - u_{tx}, \quad b = 2$$

$$Q_5 = u(u - u_{xx}) + \frac{1}{2}u^2 - u_x^2 - u_{tx}, \quad b = 3 \quad (4.106)$$

$$Q_{6\pm} = e^{\pm 2x}, \quad b = 3$$

$$Q_7 = 1 + \ln(u - u_{xx}), \quad b = 1 \quad (4.107)$$

with

$$R_{Q_1} = (1 - 1/b)((u - u_{xx})^{-2+1/b} \Delta - 2((u - u_{xx})^{-2+1/b})_x D_x), \quad b \neq 0 \quad (4.108)$$

$$R_{Q_2} = 0 \quad (4.109)$$

$$R_{Q_3} = -1, \quad b = 2 \quad (4.110)$$

$$R_{Q_4} = D_t D_x + D_x(u D_x) - 3u + u_{xx}, \quad b = 2$$

$$R_{Q_5} = D_t D_x + u D_x^2 - 3u, \quad b = 3 \quad (4.111)$$

$$R_{Q_{6\pm}} = 0, \quad b = 3$$

$$R_{Q_7} = -(u - u_{xx})^{-1} \Delta + 2((u - u_{xx})^{-1})_x D_x + ((u - u_{xx})^{-1})_{xx}, \quad b = 1 \quad (4.112)$$

given by equation (2.29). For each multiplier (4.103)–(4.107), a conserved current (4.101) can be obtained directly by either integration of the characteristic equation (4.99) or use of the scaling formula $\Psi_G(u + ptu_t, Q)$ from Proposition 2. In particular, the components of the scaling formula are given by

$$\begin{aligned} \Psi_G^t &= (u - u_{xx} + t(u_t - u_{txx}))Q, \\ \Psi_G^x &= (u + tu_t)(D_t D_x Q - D_x^2(uQ) + b D_x(u_x Q) + b(u - u_{xx})Q) \\ &\quad + (u_x + tu_{tx})(-D_t Q + D_x(uQ) - b u_x Q) + u(u - u_{xx} + t(u_t - u_{txx}))Q. \end{aligned} \quad (4.113)$$

We obtain, modulo the addition of a trivial current and an overall scaling,

$$T_1 = (u - u_{xx})^{1/b}, \quad X_1 = -bu(u - u_{xx})^{1/b}, \quad b \neq 0 \quad (4.114)$$

$$T_2 = u, \quad X_2 = \frac{1}{2}(b-1)(u^2 - u_x^2) + u(u - u_{xx}) - u_{tx} \quad (4.115)$$

$$T_3 = \frac{1}{2}(u^2 + u_x^2), \quad X_3 = u(u - u_{xx} - u_{tx}), \quad b = 2 \quad (4.116)$$

$$T_4 = \frac{1}{2}u(u^2 + u_x^2), \quad X_4 = \frac{1}{2}(u_{tx} + u(u_{xx} + \frac{3}{2}u) + \frac{1}{2}u_x^2)^2 - u_t(uu_x + \frac{1}{2}u_t), \quad b = 2 \quad (4.117)$$

$$T_5 = \frac{1}{2}u^3, \quad X_5 = \frac{1}{2}(u_{tx} + u(u_{xx} + \frac{3}{2}u) + u_x^2)^2 - \frac{1}{2}(u_t + uu_x)^2 + \frac{3}{8}u^4, \quad b = 3 \quad (4.118)$$

$$T_{6\pm} = e^{\pm 2x}u, \quad X_{6\pm} = \frac{1}{3}e^{\pm 2x}((u \mp u_x)^2 - u(u - u_{xx}) \mp 2u_x + u_{tx}), \quad b = 3 \quad (4.119)$$

$$T_7 = (u - u_{xx}) \ln(u - u_{xx}), \quad X_7 = u(u - u_{xx}) \ln(u - u_{xx}) + \frac{1}{2}(u^2 - u_x^2), \quad b = 1 \quad (4.120)$$

The first conserved current is a Hamiltonian Casimir, and the second conserved current represents conservation of mass for the b -family equation (4.88). Note these two currents coincide (modulo a trivial current) when $b = 1$. The third and fourth conserved currents respectively represent conservation of energy and momentum for the Camassa-Holm equation, while the fifth and sixth conserved currents respectively represent conservation of momentum and exponentially-weighted mass for the Degasperis-Procesi equation. The seventh conserved current represents conservation of energy.

We will now study the symmetry properties of these conservation laws (4.114)–(4.120). Consider the vector space of conserved currents

$$\begin{aligned} T &= a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4 + a_5T_5 + a_{6+}T_{6+} + a_{6-}T_{6-} + a_7T_7, \\ a_i &= \text{const.}, \quad a_1 = 0 \text{ if } b = 0, 1, \quad a_3 = a_4 = 0 \text{ if } b \neq 2, \\ a_5 &= a_6 = 0 \text{ if } b \neq 3, \quad a_7 = 0 \text{ if } b \neq 1, \end{aligned} \quad (4.121)$$

and the algebra of point symmetries

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + c_3\mathbf{X}_3 + c_4\mathbf{X}_4, \quad c_j = \text{const.}, \quad c_4 = 0 \text{ if } b \neq 0. \quad (4.122)$$

A conservation law $(D_tT + D_xX)|_{\mathcal{E}} = 0$ is homogeneous under the symmetry \mathbf{X} iff condition (3.11) is satisfied, where the characteristic of the symmetry generator is given by $P = c_1P_1 + c_2P_2 + c_3P_3 + c_4P_4$, and the multiplier for the conservation law is given by $Q = a_1Q_1 + a_2Q_2 + a_3Q_3 + a_4Q_4 + a_5Q_5 + a_{6+}Q_{6+} + a_{6-}Q_{6-} + a_7Q_7$. By using equations (4.94) and (4.108)–(4.112), we find that the condition (3.11) splits with respect to $u, u_t, u_x, u_{tx}, (u - u_{xx})^{-1+1/b}$ into a system of bilinear equations on c_j and a_i :

$$b \neq 0, 1, 2, 3: \quad a_2(c_3 - \lambda) = 0, \quad a_1c_3 = 0, \quad a_1\lambda = 0 \quad (4.123a)$$

$$b = 0: \quad c_3 - \lambda = 0 \quad (4.123b)$$

$$b = 1: \quad a_7(c_3 - \lambda) = 0, \quad (a_7 - a_2)c_3 + a_2\lambda = 0 \quad (4.123c)$$

$$b = 2: \quad a_1(c_3 - 2\lambda) = 0, \quad a_2(c_3 - \lambda) = 0, \quad a_3(2c_3 - \lambda) = 0, \quad a_4(3c_3 - \lambda) = 0 \quad (4.123d)$$

$$b = 3: \quad a_1(c_3 - 3\lambda) = 0, \quad a_2(c_3 - \lambda) = 0, \quad a_5(3c_3 - \lambda) = 0, \quad a_{6\pm}(c_3 \pm 2c_2 - \lambda) = 0 \quad (4.123e)$$

The solutions for c_j in terms of a_i determine the symmetry-homogeneity properties of the conserved current $\Phi = (T, X)$ modulo trivial currents $\Phi_{\text{triv}} = (D_x\Theta, -D_t\Theta)$. Solving the

system (4.123), we get the conditions

$$b \neq 0, 1, 2, 3 : \quad a_2 \neq 0, \quad a_1 = 0; \quad \lambda = c_3 \quad (4.124a)$$

$$a_1 \neq 0, \quad a_2 = 0; \quad c_3 = 0, \quad \lambda = 0 \quad (4.124b)$$

$$b = 0 : \quad a_2 \neq 0; \quad \lambda = c_3 \quad (4.124c)$$

$$b = 1 : \quad a_2 \neq 0, \quad a_7 = 0; \quad \lambda = c_3 \quad (4.124d)$$

$$b = 1 : \quad a_7 \neq 0, \quad a_2 = 0; \quad c_3 = 0, \quad \lambda = 0 \quad (4.124e)$$

$$b = 2 : \quad a_4 \neq 0, \quad a_2 = a_1 = a_3 = 0; \quad \lambda = 3c_3 \quad (4.124f)$$

$$b = 2 : \quad a_3 \neq 0, \quad a_2 = a_1 = a_4 = 0; \quad \lambda = 2c_3 \quad (4.124g)$$

$$b = 2 : \quad a_2^2 + a_1^2 + a_3^2 + a_4^2 \neq 0; \quad c_3 = 0, \quad \lambda = 0 \quad (4.124h)$$

$$b = 3 : \quad a_5 \neq 0, \quad a_2 = a_1 = a_{6+} = a_{6-} = 0; \quad \lambda = 3c_3 \quad (4.124i)$$

$$b = 3 : \quad a_{6\pm} \neq 0, \quad a_2 = a_1 = a_5 = a_{6\mp} = 0; \quad \lambda = c_3 \pm 2c_2 \quad (4.124j)$$

$$b = 3 : \quad a_2^2 + a_{6\pm}^2 \neq 0, \quad a_1 = a_5 = a_{6\mp} = 0; \quad c_2 = 0, \quad \lambda = c_3 \quad (4.124k)$$

$$b = 3 : \quad a_1^2 + a_{6\pm}^2 \neq 0, \quad a_2 = a_5 = a_{6\mp} = 0; \quad c_3 = \mp 3c_2, \quad \lambda = \mp c_2 \quad (4.124l)$$

$$b = 3 : \quad a_5^2 + a_{6\pm}^2 \neq 0, \quad a_2 = a_1 = a_{6\mp} = 0; \quad c_3 = \mp c_2, \quad \lambda = 3c_3 \quad (4.124m)$$

$$b = 3 : \quad a_2^2 + a_1^2 + a_5^2 + a_{6\pm}^2 \neq 0; \quad c_2 = c_3 = 0, \quad \lambda = 0 \quad (4.124n)$$

Hence, we conclude the following.

(1) For $b \neq 0$, the symmetry properties of the vector space $a_1\Phi_1 + a_2\Phi_2 + \Phi_{\text{triv}}$ consist of: (i) invariance under $\mathbf{X}_1, \mathbf{X}_2$; (ii) homogeneity of the subspace $a_2\Phi_2 + \Phi_{\text{triv}}$ under \mathbf{X}_3 with $\lambda = 1$.

(2) For $b = 0$, the symmetry properties of the vector space $a_2\Phi_2 + \Phi_{\text{triv}}$ consist of: (i) invariance under $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_4$; (ii) homogeneity under X_3 with $\lambda = 1$.

(3) For $b = 1$, the symmetry properties of the vector space $a_2\Phi_2 + a_7\Phi_7 + \Phi_{\text{triv}}$ consist of invariance under $\mathbf{X}_1, \mathbf{X}_2$.

(4) For $b = 2$, the symmetry properties of the vector space $a_1\Phi_1 + a_2\Phi_2 + a_3\Phi_3 + a_4\Phi_4 + \Phi_{\text{triv}}$ consist of: (i) invariance under $\mathbf{X}_1, \mathbf{X}_2$; (ii) homogeneity of the subspace $a_3\Phi_3 + \Phi_{\text{triv}}$ under X_3 with $\lambda = 2$; (iii) homogeneity of the subspace $a_4\Phi_4 + \Phi_{\text{triv}}$ under X_3 with $\lambda = 3$.

(5) For $b = 3$, the symmetry properties of the vector space $a_1\Phi_1 + a_2\Phi_2 + a_5\Phi_5 + a_{6+}\Phi_{6+} + a_{6-}\Phi_{6-} + \Phi_{\text{triv}}$ consist of: (i) invariance under \mathbf{X}_1 ; (ii) invariance of the subspace $a_5\Phi_5 + \Phi_{\text{triv}}$ under X_2 ; (iii) invariance of the subspaces $a_{6\pm}\Phi_{6\pm} + \Phi_{\text{triv}}$ under $\mathbf{X}_3 \mp 2\mathbf{X}_2$; (iv) homogeneity of the subspace $a_5\Phi_5 + \Phi_{\text{triv}}$ under X_3 with $\lambda = 3$; (v) homogeneity of the subspace $a_{6+}\Phi_{6+} + a_{6-}\Phi_{6-} + \Phi_{\text{triv}}$ under X_3 with $\lambda = 1$; (vi) homogeneity of the subspaces $a_{6\pm}\Phi_{6\pm} + \Phi_{\text{triv}}$ under \mathbf{X}_2 with $\lambda = \pm 2$; (vii) homogeneity of the subspaces $a_1\Phi_1 + a_{6\pm}\Phi_{6\pm} + \Phi_{\text{triv}}$ under $3X_3 \mp \mathbf{X}_2$ with $\lambda = 1$; (viii) homogeneity of the subspaces $a_5\Phi_5 + a_{6\pm}\Phi_{6\pm} + \Phi_{\text{triv}}$ under $X_3 \pm \mathbf{X}_2$ with $\lambda = 3$.

From these properties, it follows that each of the conservation laws (4.114)–(4.119) is homogeneous under the scaling symmetry \mathbf{X}_3 . Additionally, the mass conservation law (4.114) admitted for all b and the Hamiltonian Casimir conservation law (4.115) admitted for any $b \neq 0$ are invariant under the translation symmetries \mathbf{X}_1 and \mathbf{X}_2 . Similarly, the energy and momentum conservation laws (4.116)–(4.117) admitted for $b = 2$ (Camassa-Holm

equation) and the momentum conservation law (4.118) admitted for $b = 3$ (Degasperis-Procesi equation) are invariant under the space-time translation symmetries \mathbf{X}_1 and \mathbf{X}_2 , while the exponentially-weighted mass conservation law (4.119) admitted for $b = 3$ is only invariant under the time translation symmetry \mathbf{X}_1 but is homogeneous under the space translation symmetry \mathbf{X}_2 .

It is interesting to compare the formulation of conservation laws and symmetries for the b -family (4.88) and the corresponding formulation for the equivalent system (4.85). Note the b -family system (4.85) is a normal PDE system whose leading derivatives are m_t , m_x , u_{xx} . For obtaining the determining equations for symmetries and multipliers, it will be convenient to express the b -family system in solved form for the leading x -derivatives

$$m_x = -u^{-1}(m_t + bu_x m), \quad (4.125)$$

$$u_{xx} = u - m. \quad (4.126)$$

The determining equation (2.12) for infinitesimal symmetries $\mathbf{X} = P^m \partial_m + P^u \partial_u$ of the b -family system (4.125)–(4.126) is given by

$$(D_x P^m + u^{-1}(D_t P^m + bu_x P^m + bm D_x P^u) - u^{-2}(m_t + bu_x m) P^u)|_{\mathcal{E}} = 0, \quad (4.127)$$

$$(P^m - \Delta P^u)|_{\mathcal{E}} = 0, \quad (4.128)$$

where P^m and P^u are functions of t, x, u, u_x, m and t -derivatives of u, u_x, m . If P^m is eliminated in terms of P^u through equation (4.128), then equation (4.127) simplifies to the symmetry determining equation (4.89) with $P = P^u|_{m=u-u_{xx}}$. In particular, the point symmetries (4.90)–(4.91) of the b -family equation correspond to the characteristic functions

$$P_1^m = -m_t, \quad P_1^u = -u_t \quad (4.129)$$

$$P_2^m = u^{-1}(m_t + bu_x m) = -m_x, \quad P_2^u = -u_x \quad (4.130)$$

$$P_3^m = m + tm_t, \quad P_3^u = u + tu_t \quad (4.131)$$

$$P_4^m = 1 + tu^{-1}m_t = 1 - tm_x, \quad P_4^u = 1 - tu_x, \quad b = 0 \quad (4.132)$$

evaluated on the solution space of the b -family system. Each of these characteristic functions (P^m, P^u) has the form of a point symmetry generator (2.14).

The characteristic equation for conservation laws of the b -family system (4.125)–(4.126) is given by

$$D_t T + D_x X = Q^m G^m + Q^u G^u, \quad G^m = m_x + u^{-1}(m_t + bu_x m), \quad G^u = u_{xx} - u + m \quad (4.133)$$

with the multiplier

$$Q = (Q^m, Q^u), \quad Q^m = E_m(X), \quad Q^u = E_{u_x}(X) \quad (4.134)$$

where X is a function of t, x, u, u_x, m , and t -derivatives of u, u_x, m . Consequently, multipliers Q are also functions only of t, x, u, u_x, m , and t -derivatives of u, u_x, m . Since the b -family system is of second-order, its low-order conservation laws are defined by multipliers with a first-order form

$$Q^m(t, x, u, u_x, m), \quad Q^u(t, x, u, u_x, m) \quad (4.135)$$

where u_t is excluded because the only second-order leading derivative in the system (4.125)–(4.126) is u_{xx} which cannot be obtained from u_t by differentiations. The corresponding low-order conserved currents will then have the form

$$\Phi = (T(t, x, u, u_x, m), X(t, x, u, u_x, m)). \quad (4.136)$$

Thus, the low-order conservation laws admitted by the b -family system is a strict subset of the low-order conservation laws (4.101) admitted by the b -family equation (4.88).

The determining system (2.28) and (2.32) for low-order conservation laws (4.136) of the b -family system (4.125)–(4.126) consists of only the adjoint-symmetry equations

$$(Q^u - Q^m - D_t(u^{-1}Q^m) + u^{-1}bu_xQ^m)|_{\mathcal{E}} = 0, \quad (4.137)$$

$$(-\Delta Q^u - bu^{-1}D_x(mQ^m) - u^{-2}(bu_xm + m_t)Q^m)|_{\mathcal{E}} = 0 \quad (4.138)$$

since the Helmholtz equations (2.32) can be shown to hold identically due to Q not containing any t -derivatives of u . Hence, low-order adjoint-symmetries of the b -family system are the same as low-order multipliers.

If Q^u is eliminated in terms of Q^m through equation (4.137), then equation (4.138) simplifies to the adjoint-symmetry determining equation (4.102) for the b -family equation, with $Q = u^{-1}Q^u|_{m=u-u_{xx}}$. In particular, the low-order multipliers (4.103)–(4.106) admitted by the b -family equation correspond to the multipliers

$$Q_1^m = um^{-1+1/b}, \quad Q_1^u = 0, \quad b \neq 0 \quad (4.139)$$

$$Q_2^m = u, \quad Q_2^u = (1-b)u_x, \quad b \neq 1 \quad (4.140)$$

$$Q_3^m = u^2, \quad Q_3^u = u_t, \quad b = 2 \quad (4.141)$$

$$Q_4^m = u^2m + \frac{1}{2}u(u^2 - u_x^2) - uu_{tx}, \quad Q_4^u = \frac{1}{2}u_x(u_x^2 - u^2) + (u_t - uu_x)m + um_t - u_{ttx}, \quad b = 2 \quad (4.142)$$

$$Q_5^m = u^2m + \frac{1}{2}u^3 - uu_x^2 - uu_{tx}, \quad Q_5^u = 2u_x(u_x^2 - u^2) + (u_t - 2uu_x)m + um_t - u_{ttx}, \quad b = 3 \quad (4.143)$$

$$Q_{6\pm}^m = e^{\pm 2x}u, \quad Q_{6\pm}^u = -2e^{\pm 2x}(u + u_x), \quad b = 3 \quad (4.144)$$

$$Q_7^m = u(1 + \ln(m)), \quad Q_7^u = -u_x, \quad b = 1 \quad (4.145)$$

for the b -family system. Note that only the multipliers (4.139), (4.140), (4.144), (4.145) have the low-order form (4.135). Hence, the other multipliers (4.141), (4.142), (4.143) are higher-order multipliers for the b -family system.

This shows that conservation laws of higher-order for a PDE system can be equivalent to low-order conservation laws for an equivalent single PDE.

4.5. Navier-Stokes equations. Our final example is the Navier-Stokes equations [21] for compressible, viscous fluids in two dimensions

$$\rho_t + \nabla \cdot (\rho \vec{u}) = 0, \quad (4.146)$$

$$(\rho \vec{u})_t + \nabla \cdot (\rho \vec{u} \odot \vec{u} + p \mathbf{I}) = \nabla \cdot \boldsymbol{\sigma}, \quad (4.147)$$

$$\boldsymbol{\sigma} = \mu(\nabla \odot \vec{u} - \frac{1}{2}\nabla \cdot \vec{u} \mathbf{I}) \quad (4.148)$$

where ρ is the density and $\vec{u} = (u^1, u^2)$ is the velocity, which are functions of t and $\vec{x} = (x, y)$, and where p is the pressure, μ is the viscosity, and $\boldsymbol{\sigma}$ is the trace-free stress tensor. Here \odot denotes the symmetric product of vectors, and \mathbf{I} denotes the 2x2 identity matrix. In general, p and μ are functions of ρ . Unlike the non-viscous equations, this system has no Hamiltonian structure.

We will consider the slightly-compressible case, in which μ is constant and p is linear in ρ . This yields the PDE system

$$\rho_t + (\rho u^1)_x + (\rho u^2)_y = 0, \quad (4.149a)$$

$$u_t^1 + u^1 u_x^1 + u^2 u_y^1 = (1/\rho)(-\kappa \rho_x + \mu(u_{xx}^1 + u_{yy}^1)), \quad (4.149b)$$

$$u_t^2 + u^1 u_x^2 + u^2 u_y^2 = (1/\rho)(-\kappa \rho_y + \mu(u_{xx}^2 + u_{yy}^2)), \quad (4.149c)$$

$$\mu = \text{const.} \neq 0, \quad \kappa = \text{const.} \neq 0. \quad (4.149d)$$

Note that ρ_t, ρ_x, ρ_y and $u_t^1, u_t^2, u_{xx}^1, u_{yy}^1, u_{xx}^2, u_{yy}^2$ are leading derivatives in the system (4.149), so it is a normal PDE system.

The determining system (2.12) for infinitesimal symmetries $\mathbf{X} = P^\rho \partial_\rho + P^{u^1} \partial_{u^1} + P^{u^2} \partial_{u^2}$ of the Navier-Stokes system (4.149) is given by

$$(D_t P^\rho + D_x(u^1 P^\rho + \rho P^{u^1}) + D_y(u^2 P^\rho + \rho P^{u^2}))|_{\mathcal{E}} = 0, \quad (4.150a)$$

$$\begin{aligned} (D_t P^{u^1} + u^1 D_x P^{u^1} + u^2 D_y P^{u^1} + u_x^1 P^{u^1} + u_y^1 P^{u^2} + \kappa D_x(\rho^{-1} P^\rho) \\ - \mu \rho^{-1}(D_x^2 P^{u^1} + D_y^2 P^{u^1}) + \mu \rho^{-2}(u_{xx}^1 + u_{yy}^1) P^\rho)|_{\mathcal{E}} = 0, \end{aligned} \quad (4.150b)$$

$$\begin{aligned} (D_t P^{u^2} + u^1 D_x P^{u^2} + u^2 D_y P^{u^2} + u_x^2 P^{u^1} + u_y^2 P^{u^2} + \kappa D_y(\rho^{-1} P^\rho) \\ - \mu \rho^{-1}(D_x^2 P^{u^2} + D_y^2 P^{u^2}) + \mu \rho^{-2}(u_{xx}^2 + u_{yy}^2) P^\rho)|_{\mathcal{E}} = 0. \end{aligned} \quad (4.150c)$$

Off of the solution space \mathcal{E} , these equations (4.150) take the form

$$\begin{aligned} G^{\rho'}(P^\rho, P^{u^1}, P^{u^2}) &= R_P^\rho(G^\rho, G^{u^1}, G^{u^2}) \\ G^{u^1'}(P^\rho, P^{u^1}, P^{u^2}) &= R_P^{u^1}(G^\rho, G^{u^1}, G^{u^2}) \\ G^{u^2'}(P^\rho, P^{u^1}, P^{u^2}) &= R_P^{u^2}(G^\rho, G^{u^1}, G^{u^2}) \end{aligned} \quad (4.151)$$

as given by equation (3.4), with

$$\begin{aligned} G^\rho &= \rho_t + (\rho u^1)_x + (\rho u^2)_y \\ G^{u^1} &= u_t^1 + u^1 u_x^1 + u^2 u_y^1 + (1/\rho)(\kappa \rho_x - \mu(u_{xx}^1 + u_{yy}^1)) \\ G^{u^2} &= u_t^2 + u^1 u_x^2 + u^2 u_y^2 + (1/\rho)(\kappa \rho_y - \mu(u_{xx}^2 + u_{yy}^2)). \end{aligned} \quad (4.152)$$

A direct calculation of characteristic functions $P = (P^\rho, P^{u^1}, P^{u^2})$ for point symmetries yields

$$P_1 = (-\rho_x, -u_x^1, -u_x^2), \quad P_2 = (-\rho_y, -u_y^1, -u_y^2), \quad P_3 = (-\rho_t, -u_t^1, -u_t^2), \quad (4.153)$$

$$P_4 = (-t\rho_x, 1 - tu_x^1, -tu_x^2), \quad P_5 = (-t\rho_y, -tu_y^1, 1 - tu_y^2), \quad (4.154)$$

$$P_6 = (y\rho_x - x\rho_y, yu_x^1 - xu_y^1 - u^2, yu_x^2 - xu_y^2 + u^1), \quad (4.155)$$

$$P_7 = (-t\rho_t - x\rho_x - y\rho_y - \rho, -tu_t^1 - xu_x^1 - yu_y^1, -tu_t^2 - xu_x^2 - yu_y^2), \quad (4.156)$$

with

$$R_{P_1} = \begin{pmatrix} -D_x & 0 & 0 \\ 0 & -D_x & 0 \\ 0 & 0 & -D_x \end{pmatrix}, R_{P_2} = \begin{pmatrix} -D_y & 0 & 0 \\ 0 & -D_y & 0 \\ 0 & 0 & -D_y \end{pmatrix}, R_{P_3} = \begin{pmatrix} -D_t & 0 & 0 \\ 0 & -D_t & 0 \\ 0 & 0 & -D_t \end{pmatrix}, \quad (4.157)$$

$$R_{P_4} = \begin{pmatrix} -tD_x & 0 & 0 \\ 0 & -tD_x & 0 \\ 0 & 0 & -tD_x \end{pmatrix}, R_{P_5} = \begin{pmatrix} -tD_y & 0 & 0 \\ 0 & -tD_y & 0 \\ 0 & 0 & -tD_y \end{pmatrix}, \quad (4.158)$$

$$R_{P_6} = \begin{pmatrix} yD_x - xD_y & 0 & 0 \\ 0 & yD_x - xD_y & -1 \\ 0 & 1 & yD_x - xD_y \end{pmatrix}, \quad (4.159)$$

$$R_{P_7} = \begin{pmatrix} -tD_t - xD_x - yD_y - 2 & 0 & 0 \\ 0 & -tD_t - xD_x - yD_y - 1 & 0 \\ 0 & 0 & -tD_t - xD_x - yD_y - 1 \end{pmatrix} \quad (4.160)$$

where $R_P = (R_P^\rho, R_P^{u^1}, R_P^{u^2})$ is the matrix defined by factoring out the column vector $(G^\rho, G^{u^1}, G^{u^2})^t$ in equation (4.151).

From this calculation, it follows that all point symmetries of the Navier-Stokes system (4.149) are generated by translations in x, y, t

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = \partial_y, \quad \mathbf{X}_3 = \partial_t, \quad (4.161)$$

Galilean boosts with respect to x, y

$$\mathbf{X}_4 = \partial_{u^1} + t\partial_x, \quad \mathbf{X}_5 = \partial_{u^2} + t\partial_y, \quad (4.162)$$

a rotation

$$\mathbf{X}_6 = -u^2\partial_{u^1} + u^1\partial_{u^2} - y\partial_x + x\partial_y, \quad (4.163)$$

and a scaling

$$\mathbf{X}_7 = -\rho\partial_\rho + t\partial_t + x\partial_x + y\partial_y. \quad (4.164)$$

Their respective group actions consist of

$$x \rightarrow x + \epsilon, \quad (4.165)$$

$$y \rightarrow y + \epsilon, \quad (4.166)$$

$$t \rightarrow t + \epsilon, \quad (4.167)$$

$$x \rightarrow x + \epsilon t, \quad u^1 \rightarrow u^1 + \epsilon \quad (4.168)$$

$$y \rightarrow y + \epsilon t, \quad u^2 \rightarrow u^2 + \epsilon \quad (4.169)$$

$$x \rightarrow \cos(\epsilon)x - \sin(\epsilon)y, \quad y \rightarrow \cos(\epsilon)y + \sin(\epsilon)x, \quad (4.170)$$

$$u^1 \rightarrow \cos(\epsilon)u^1 - \sin(\epsilon)u^2, \quad u^2 \rightarrow \cos(\epsilon)u^2 + \sin(\epsilon)u^1,$$

$$x \rightarrow e^\epsilon x, \quad y \rightarrow e^\epsilon y, \quad t \rightarrow e^\epsilon t, \quad \rho \rightarrow e^{-\epsilon} \rho \quad (4.171)$$

in terms of a parameter ϵ .

To formulate the characteristic equation for conservation laws of the Navier-Stokes system (4.149), it is simplest to use the solved form in terms of the leading derivatives ρ_t, u_t^1, u_t^2 . This yields

$$D_t T + D_x X + D_y Y = (Q^\rho, Q^{u^1}, Q^{u^2})(G^\rho, G^{u^1}, G^{u^2})^t \quad (4.172)$$

with the multiplier $Q = (Q^\rho, Q^{u^1}, Q^{u^2})$ given by

$$Q^\rho = E_\rho(T), \quad Q^{u^1} = E_{u^1}(T), \quad Q^{u^2} = E_{u^2}(T) \quad (4.173)$$

where T is a function of t, x, y, ρ, u^1, u^2 , and x, y -derivatives of ρ, u^1, u^2 , with all t -derivatives being eliminated from T through the PDEs in the system (4.149). Consequently, multipliers $Q = (Q^\rho, Q^{u^1}, Q^{u^2})$ are also functions only of t, x, y, ρ, u^1, u^2 , and x, y -derivatives of ρ, u^1, u^2 . Low-order conservation laws are defined by multipliers with the first-order form

$$Q = Q(t, x, y, \rho, u^1, u^2, u_x^1, u_y^1, u_x^2, u_y^2). \quad (4.174)$$

The corresponding conserved currents $\Phi = (T, X, Y)$ will have the form

$$\Phi = \Phi(t, x, y, \rho, u^1, u^2, u_x^1, u_y^1, u_x^2, u_y^2) \quad (4.175)$$

as determined by equations (4.172) and (4.173). For multipliers and currents of this form, it is straightforward to show that their dependence on derivatives of u^1, u^2 must be at most linear, from splitting the equations with respect to derivatives of ρ, u^1, u^2 , and using the linearity of the Navier-Stokes system (4.149) in second-order derivatives of u^1, u^2 .

The determining system (2.28) and (2.32) for all low-order conservation laws (4.175) consists of the adjoint-symmetry equations

$$\begin{aligned} & (-D_t Q^\rho - u^1 D_x Q^\rho - u^2 D_y Q^\rho - \kappa \rho^{-1} (D_x Q^{u^1} + D_y Q^{u^2}) \\ & + \mu \rho^{-2} ((u_{xx}^1 + u_{yy}^1) Q^{u^1} + (u_{xx}^2 + u_{yy}^2) Q^{u^2}))|_{\mathcal{E}} = 0, \end{aligned} \quad (4.176a)$$

$$\begin{aligned} & (-D_t Q^{u^1} - \rho D_x Q^\rho + u_x^2 Q^{u^2} - u^1 D_x Q^{u^1} - D_y (u^2 Q^{u^1}) \\ & - \mu (D_x^2 (\rho^{-1} Q^{u^1}) + D_y^2 (\rho^{-1} Q^{u^1})))|_{\mathcal{E}} = 0, \end{aligned} \quad (4.176b)$$

$$\begin{aligned} & (-D_t Q^{u^2} - \rho D_y Q^\rho + u_y^1 Q^{u^1} - u^2 D_y Q^{u^2} - D_x (u^1 Q^{u^2}) \\ & - \mu (D_x^2 (\rho^{-1} Q^{u^2}) + D_y^2 (\rho^{-1} Q^{u^2})))|_{\mathcal{E}} = 0, \end{aligned} \quad (4.176c)$$

and the Helmholtz equations

$$Q_{u_x^1}^\rho = 0, \quad Q_{u_y^1}^\rho = 0, \quad Q_{u_x^2}^\rho = 0, \quad Q_{u_y^2}^\rho = 0, \quad (4.177a)$$

$$Q_{u_x^1}^{u^1} = 0, \quad Q_{u_y^1}^{u^1} = 0, \quad Q_{u_x^2}^{u^1} = 0, \quad Q_{u_y^2}^{u^1} = 0, \quad (4.177b)$$

$$Q_{u_x^1}^{u^2} = 0, \quad Q_{u_y^1}^{u^2} = 0, \quad Q_{u_x^2}^{u^2} = 0, \quad Q_{u_y^2}^{u^2} = 0, \quad (4.177c)$$

after simplifications. Thus, all low-order multipliers consist of adjoint-symmetries that have no dependence on derivatives of ρ, u^1, u^2 . Because the Navier-Stokes system (4.149) has no Hamiltonian structure, there is no correspondence between multipliers and symmetry characteristics.

A straightforward computation of adjoint-symmetries

$$Q = (Q^\rho(t, x, y, u^1, u^2), Q^{u^1}(t, x, y, u^1, u^2), Q^{u^2}(t, x, y, u^1, u^2)) \quad (4.178)$$

yields

$$Q_1 = (1, 0, 0), \quad (4.179)$$

$$Q_2 = (u^1, \rho, 0), \quad Q_3 = (u^2, 0, \rho), \quad (4.180)$$

$$Q_4 = (tu^1 - x, t\rho, 0), \quad Q_5 = (tu^2 - y, 0, t\rho), \quad (4.181)$$

$$Q_6 = (xu^2 - yu^1, -y\rho, x\rho), \quad (4.182)$$

with

$$R_{Q_1} = 0, \quad (4.183)$$

$$R_{Q_2} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_{Q_3} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad (4.184)$$

$$R_{Q_4} = \begin{pmatrix} 0 & -t & 0 \\ -t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_{Q_5} = \begin{pmatrix} 0 & 0 & -t \\ 0 & 0 & 0 \\ -t & 0 & 0 \end{pmatrix}, \quad (4.185)$$

$$R_{Q_6} = \begin{pmatrix} 0 & y & -x \\ y & 0 & 0 \\ -x & 0 & 0 \end{pmatrix} \quad (4.186)$$

where $R_Q = (R_Q^\rho, R_Q^{u^1}, R_Q^{u^2})$ is the matrix defined by factoring out the column vector $(G^\rho, G^{u^1}, G^{u^2})^t$ in the form of the adjoint-symmetry equations off of the solution space \mathcal{E} ,

$$\begin{aligned} G^{\rho\prime*}(Q^\rho, Q^{u^1}, Q^{u^2}) &= R_Q^\rho(G^\rho, G^{u^1}, G^{u^2}) \\ G^{u^1\prime*}(Q^\rho, Q^{u^1}, Q^{u^2}) &= R_Q^{u^1}(G^\rho, G^{u^1}, G^{u^2}) \\ G^{u^2\prime*}(Q^\rho, Q^{u^1}, Q^{u^2}) &= R_Q^{u^2}(G^\rho, G^{u^1}, G^{u^2}) \end{aligned} \quad (4.187)$$

as given by equation (3.4).

For each multiplier (4.179)–(4.182), a conserved current (4.175) can be obtained directly by either integration of the characteristic equation (4.172) or use of the scaling formula

$\Psi_G((-t\rho_t - x\rho_x - y\rho_y - \rho, -tu_t^1 - xu_x^1 - yu_y^2, -tu_t^2 - xu_x^2 - yu_y^2), (Q^\rho, Q^{u^1}, Q^{u^2}))$ from Proposition 2. In particular, the components of the scaling formula are given by

$$\begin{aligned}
\Psi_G^t &= -(t\rho_t + x\rho_x + y\rho_y + \rho)Q^\rho - (tu_t^1 + xu_x^1 + yu_y^1)Q^{u^1} - (tu_t^2 + xu_x^2 + yu_y^2)Q^{u^2}, \\
\Psi_G^x &= -((tu_t^1 + xu_x^1 + yu_y^1)\rho + (t\rho_t + x\rho_x + y\rho_y + \rho)u^1)Q^\rho - (tu_t^2 + xu_x^2 + yu_y^2)u^1Q^{u^2} \\
&\quad - (\kappa(t\rho_t + x\rho_x + y\rho_y + \rho)\rho^{-1} + (tu_t^1 + xu_x^1 + yu_y^1)u^1)Q^{u^1} \\
&\quad + \mu(tu_{tx}^1 + xu_{xx}^1 + yu_{xy}^1 + u_x^1)\rho^{-1}Q^{u^1} + \mu(tu_{tx}^2 + xu_{xx}^2 + yu_{xy}^2 + u_x^2)\rho^{-1}Q^{u^2} \\
&\quad - \mu(tu_t^1 + xu_x^1 + yu_y^1)D_x(\rho^{-1}Q^{u^1}) - \mu(tu_t^2 + xu_x^2 + yu_y^2)D_x(\rho^{-1}Q^{u^2}), \\
\Psi_G^y &= -((tu_t^2 + xu_x^2 + yu_y^2)\rho + (t\rho_t + x\rho_x + y\rho_y + \rho)u^2)Q^\rho - (tu_t^1 + xu_x^1 + yu_y^1)u^2Q^{u^1} \\
&\quad - (\kappa(t\rho_t + x\rho_x + y\rho_y + \rho)\rho^{-1} + (tu_t^2 + xu_x^2 + yu_y^2)u^2)Q^{u^2} \\
&\quad + \mu(tu_{ty}^1 + xu_{xy}^1 + yu_{yy}^1 + u_y^1)\rho^{-1}Q^{u^1} + \mu(tu_{ty}^2 + xu_{xy}^2 + yu_{yy}^2 + u_y^2)\rho^{-1}Q^{u^2} \\
&\quad - \mu(tu_t^1 + xu_x^1 + yu_y^1)D_y(\rho^{-1}Q^{u^1}) - \mu(tu_t^2 + xu_x^2 + yu_y^2)D_y(\rho^{-1}Q^{u^2}).
\end{aligned} \tag{4.188}$$

We obtain, modulo the addition of a trivial current and an overall scaling,

$$T_1 = \rho, \quad X_1 = \rho u^1, \quad Y_1 = \rho u^2 \tag{4.189}$$

$$T_2 = \rho u^1 \quad X_2 = \rho(\kappa + (u^1)^2) - \mu u_x^1, \quad Y_2 = \rho u^1 u^2 - \mu u_y^1 \tag{4.190}$$

$$T_3 = \rho u^2 \quad X_3 = \rho u^1 u^2 - \mu u_x^2, \quad Y_3 = \rho(\kappa + (u^2)^2) - \mu u_y^2 \tag{4.191}$$

$$T_4 = (tu^1 - x)\rho, \quad X_4 = \rho t(\kappa + (u^1)^2) - x\rho u^1 - \mu tu_x^1, \quad Y_4 = \rho tu^1 u^2 - x\rho u^2 - \mu tu_y^1 \tag{4.192}$$

$$T_5 = (tu^2 - y)\rho, \quad X_5 = \rho tu^1 u^2 - y\rho u^1 - \mu tu_x^2, \quad Y_5 = \rho t(\kappa + (u^2)^2) - y\rho u^2 - \mu tu_y^2, \tag{4.193}$$

$$\begin{aligned}
T_6 &= (xu^2 - yu^1)\rho, \quad X_6 = (xu^2 - yu^1)\rho u^1 - \kappa y\rho + \mu(yu_x^1 - xu_x^2 + u^2), \\
Y_6 &= (xu^2 - yu^1)\rho u^2 + \kappa x\rho + \mu(yu_y^1 - xu_y^2 - u^1)
\end{aligned} \tag{4.194}$$

The first conserved current represents conservation of mass. The second and third conserved currents represent conservation of momentum, while the fourth and fifth conserved currents represent conservation of Galilean momentum. The sixth conserved current represents conservation of angular momentum.

We will now study the symmetry properties of these conservation laws (4.189)–(4.194). Consider the vector space of conserved currents

$$T = a_1 T_1 + a_2 T_2 + a_3 T_3 + a_4 T_4 + a_5 T_5 + a_6 T_6, \quad a_i = \text{const.}, \tag{4.195}$$

and the algebra of point symmetries

$$\mathbf{X} = c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 + c_3 \mathbf{X}_3 + c_4 \mathbf{X}_4 + c_5 \mathbf{X}_5 + c_6 \mathbf{X}_6 + c_7 \mathbf{X}_7, \quad c_j = \text{const.} \tag{4.196}$$

A conservation law $(D_t T + D_x X + D_y Y)|_{\mathcal{E}} = 0$ is homogeneous under the symmetry \mathbf{X} iff condition (3.11) is satisfied, where the characteristic of the symmetry generator is given by $P = c_1 P_1 + c_2 P_2 + c_3 P_3 + c_4 P_4 + c_5 P_5 + c_6 P_6 + c_7 P_7$, and the multiplier for the conservation law is given by $Q = a_1 Q_1 + a_2 Q_2 + a_3 Q_3 + a_4 Q_4 + a_5 Q_5 + a_6 Q_6$. By using equations (4.157)–(4.160) and (4.183)–(4.186), we find that the condition (3.11) splits into a system of bilinear

equations on c_j and a_i :

$$\begin{aligned} a_6(2c_7 - \lambda) &= 0, & a_5(2c_7 - \lambda) - a_4c_6 + a_6c_4 &= 0, & a_4(2c_7 - \lambda) + a_5c_6 - a_6c_5 &= 0, \\ a_3(c_7 - \lambda) - a_2c_6 + a_5c_3 + a_6c_1 &= 0, & a_2(c_7 - \lambda) + a_3c_6 + a_4c_3 - a_6c_2 &= 0, \\ a_1(c_7 - \lambda) + a_2c_4 + a_3c_5 - a_4c_1 - a_5c_2 &= 0. \end{aligned} \quad (4.197)$$

The solutions for c_j in terms of a_i determine the symmetry-homogeneity properties of the conserved current $\Phi = (T, X, Y)$ modulo trivial currents $\Phi_{\text{triv}} = (D_x\Theta^1 + D_y\Theta^2, -D_t\Theta^1 + D_y\Theta^3, -D_t\Theta^2 - D_x\Theta^3)$. By considering the subspaces generated by $\{a_i\}$, solving the system (4.197) in each case, and merging the solutions, we get the conditions

$$a_1 \neq 0, \quad a_2 = a_3 = a_4 = a_5 = a_6 = 0; \quad \lambda = c_7 \quad (4.198a)$$

$$a_1^2 + a_2^2 + a_3^2 \neq 0, \quad a_4 = a_5 = a_6 = 0; \quad \lambda = c_7, \quad c_6 = 0, \quad a_2c_4 + a_3c_5 = 0 \quad (4.198b)$$

$$\begin{aligned} a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 &\neq 0, \quad a_1a_6 - a_2a_5 + a_3a_4 = 0; \\ \lambda = 2c_7 \neq 0, \quad a_2c_6 + a_3c_7 - a_5c_3 - a_6c_1 &= 0, \quad a_2c_7 - a_3c_6 - a_4c_3 + a_6c_2 = 0, \end{aligned} \quad (4.198c)$$

$$\begin{aligned} a_4c_6 - a_6c_4 &= 0, \quad a_5c_6 - a_6c_5 = 0 \\ a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 &\neq 0; \\ \lambda = 0, \quad c_7 = 0, \quad a_2c_6 - a_5c_3 - a_6c_1 &= 0, \quad a_3c_6 + a_4c_3 - a_6c_2 = 0, \\ a_4c_6 - a_6c_4 &= 0, \quad a_5c_6 - a_6c_5 = 0 \end{aligned} \quad (4.198d)$$

Hence, we conclude the following.

(1) The symmetry properties of the vector space $a_1\Phi_1 + a_2\Phi_2 + a_3\Phi_3 + a_4\Phi_4 + a_5\Phi_5 + a_6\Phi_6 + \Phi_{\text{triv}}$ for arbitrary a_i consist of invariance under $a_5\mathbf{X}_1 - a_4\mathbf{X}_2 - a_6\mathbf{X}_3$ and $a_2\mathbf{X}_1 + a_3\mathbf{X}_2 + a_4\mathbf{X}_4 + a_5\mathbf{X}_5 + a_6\mathbf{X}_6$.

(2) The only additional symmetry properties consist of: (i) invariance of the subspace $a_1\Phi_1 + \Phi_{\text{triv}}$ under $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5, \mathbf{X}_6$; (ii) invariance of the subspace $a_1\Phi_1 + a_2\Phi_2 + a_3\Phi_3 + \Phi_{\text{triv}}$ under $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, a_3\mathbf{X}_4 - a_2\mathbf{X}_5$; (iii) homogeneity of the subspaces $a_1\Phi_1 + \Phi_{\text{triv}}$ and $a_1\Phi_1 + a_2\Phi_2 + a_3\Phi_3 + \Phi_{\text{triv}}$ under \mathbf{X}_7 with $\lambda = 1$; (iv) homogeneity of the projective subspace $a_1\Phi_1 + a_2\Phi_2 + a_3\Phi_3 + a_4\Phi_4 + a_5\Phi_5 + a_6\Phi_6 + \Phi_{\text{triv}}$, $a_1a_6 - a_2a_5 + a_3a_4 = 0$, under $a_1\mathbf{X}_1 + a_2\mathbf{X}_3 + a_4\mathbf{X}_7$ with $\lambda = 2a_4$, and $a_1\mathbf{X}_2 + a_3\mathbf{X}_3 + a_5\mathbf{X}_7$ with $\lambda = 2a_5$, and $a_3\mathbf{X}_1 - a_2\mathbf{X}_2 + a_6\mathbf{X}_7$ with $\lambda = 2a_6$;

From these properties, it follows that each of the conservation laws (4.189)–(4.194) is homogeneous under the scaling symmetry \mathbf{X}_7 . Additionally, all six conservation laws are invariant under the time-translation symmetry \mathbf{X}_3 , while the mass conservation law (4.189) and the momentum conservation laws (4.190)–(4.191) are invariant under the space-translation symmetries $\mathbf{X}_1, \mathbf{X}_2$. Also, the Galilean momentum conservation laws (4.192)–(4.193) are invariant under both of the Galilean boost symmetries $\mathbf{X}_4, \mathbf{X}_5$, and the angular momentum conservation law (4.194) is invariant under the rotation symmetry \mathbf{X}_6 .

5. CONCLUDING REMARKS

If a normal PDE system is an Euler-Lagrange system or a Hamiltonian system, then there is a direct correspondence between conservation laws and variational symmetries or Hamiltonian symmetries. In these cases, symmetry invariance of conservation laws is connected to abelian subalgebras in the symmetry algebra of the PDE system, as illustrated by the examples in Sec. 4.2 and Sec. 4.4. We will explore this connection in detail elsewhere.

In contrast, when a normal PDE system does not have any Lagrangian or Hamiltonian formulation, conservation laws instead correspond to adjoint-symmetries that satisfy conditions for being multipliers, and there is no obvious relationship between symmetry invariance of conservation laws and properties of the symmetry algebra of the PDE system. However, it may still be fruitful to look for mappings from symmetries and into multipliers, and vice versa, in which case the symmetry properties of conservation laws may be related to properties of the symmetry algebra itself. This is a promising direction for future work.

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